

# Stability of interval positive continuous-time linear systems

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**Abstract.** It is shown that the convex linear combination of the Hurwitz polynomials of positive linear systems is also the Hurwitz polynomial. The Kharitonov theorem is extended to the positive interval linear systems. It is also shown that the interval positive linear system described by state equation  $\dot{x} = Ax$ ,  $A \in \mathfrak{R}^{n \times n}$ ,  $A_1 \leq A \leq A_2$  is asymptotically stable if and only if the matrices  $A_k$ ,  $k = 1, 2$  are Hurwitz Metzler matrices.

**Key words:** interval, positive, linear, continuous-time, system, stability.

## 1. Introduction

A dynamical system is called positive if its state variables take nonnegative values for all nonnegative inputs and nonnegative initial conditions. The positive linear systems have been investigated in [1, 2] and positive nonlinear systems in [3–7]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems with different fractional orders have been addressed in [8–13]. Descriptor (singular) linear systems have been analyzed in [1, 5, 14] and the stability of a class of nonlinear fractional-order systems in [3, 13]. Application of Drazin inverse to analysis of descriptor fractional discrete-time linear systems has been presented in [15] and stability of discrete-time switched systems with unstable subsystems in [16]. The robust stabilization of discrete-time positive switched systems with uncertainties has been addressed in [17]. Comparison of three methods of analysis of the descriptor fractional systems has been presented in [18]. Stability of linear fractional order systems with delays has been analyzed in [19] and simple conditions for practical stability of positive fractional systems have been proposed in [20].

In this paper the asymptotic stability of interval positive continuous-time linear systems will be investigated.

The paper is organized as follows. In Section 2 some basic definitions and theorems concerning positive linear systems and polynomials with interval coefficients are recalled. The convex linear combination of Hurwitz polynomials of positive linear systems is addressed in Section 3. An extension of the Kharitonov theorem to positive interval linear systems is presented

in Section 4, the stability of interval positive linear systems described by the state equation is presented in Section 4 and the concluding remarks are given in Section 5.

The following notations will be used:  $\mathfrak{R}$  – the set of real numbers,  $\mathfrak{R}^{n \times m}$  – the set of  $n \times m$  real matrices,  $\mathfrak{R}_+^{n \times m}$  – the set of  $n \times m$  real matrices with nonnegative entries and  $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$ ,  $M_n$  – the set of  $n \times n$  Metzler matrices (real matrices with nonnegative off-diagonal entries),  $I_n$  – the  $n \times n$  identity matrix, for  $A = [a_{ij}] \in \mathfrak{R}^{n \times n}$  and  $B = [b_{ij}] \in \mathfrak{R}^{n \times n}$  inequality  $A \geq B$  means  $a_{ij} \geq b_{ij}$  for  $i, j = 1, 2, \dots, n$ .

## 2. Preliminaries

Consider the autonomous continuous-time linear system

$$\dot{x}(t) = Ax(t), \quad t \geq 0, \quad (1)$$

where  $x(t) \in \mathfrak{R}^n$  is the state vector and  $A \in \mathfrak{R}^{n \times n}$ .

**Definition 1.** [2, 21] The system (1) is called positive if  $x(t) \in \mathfrak{R}_+^n$ ,  $t \geq 0$  for any initial conditions  $x_0 = x(0) \in \mathfrak{R}_+^n$ .

**Theorem 1.** [2, 21] The system (1) is positive if and only if its matrix  $A$  is the Metzler matrix (off-diagonal entries are nonnegative).

**Definition 2.** [2, 21] The positive system (1) is called asymptotically stable if

$$\lim_{t \rightarrow \infty} x(t) = 0 \text{ for all } x(0) \in \mathfrak{R}_+^n.$$

**Theorem 2.** [2, 21, 22] The positive system (1) is asymptotically stable if and only if one of the equivalent conditions is satisfied:

1) All coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 \quad (2)$$

are positive, i.e.  $a_k > 0$  for  $k = 0, 1, \dots, n - 1$ .

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2) All principal minors  $\bar{M}_i, i = 1, \dots, n$  of the matrix  $-A$  are positive, i.e.

$$\bar{M}_1 = |-a_{11}| > 0, \bar{M}_1 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \dots, \quad (3)$$

$$\bar{M}_n = \det[-A] > 0.$$

3) There exists strictly positive vector  $\lambda^T = [\lambda_1, \dots, \lambda_n]^T, \lambda_k > 0, k = 1, \dots, n$  such that

$$A\lambda < 0 \text{ or } A^T\lambda < 0. \quad (4)$$

If  $\det A \neq 0$  then we may choose  $\lambda = -A^{-1}c$ , where  $c \in \mathbb{R}^n$  is any strictly positive vector.

Consider the set (family) of the  $n$ -degree polynomials

$$p_n(s) := a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 \quad (5a)$$

with the interval coefficients

$$\underline{a}_i \leq a_i \leq \bar{a}_i, \quad i = 0, 1, \dots, n. \quad (5b)$$

Using (5) we define the following four polynomials:

$$\begin{aligned} p_{1n}(s) &:= \underline{a}_0 + \underline{a}_1 s + \bar{a}_2 s^2 + \bar{a}_3 s^3 + \underline{a}_4 s^4 + \underline{a}_5 s^5 + \dots \\ p_{2n}(s) &:= \underline{a}_0 + \bar{a}_1 s + \bar{a}_2 s^2 + \underline{a}_3 s^3 + \underline{a}_4 s^4 + \bar{a}_5 s^5 + \dots \\ p_{3n}(s) &:= \bar{a}_0 + \underline{a}_1 s + \underline{a}_2 s^2 + \bar{a}_3 s^3 + \bar{a}_4 s^4 + \underline{a}_5 s^5 + \dots \\ p_{4n}(s) &:= \bar{a}_0 + \bar{a}_1 s + \underline{a}_2 s^2 + \underline{a}_3 s^3 + \bar{a}_4 s^4 + \bar{a}_5 s^5 + \dots \end{aligned} \quad (6)$$

**Kharitonov theorem:** The set of polynomials (5) is asymptotically stable if and only if the four polynomials (6) are asymptotically stable.

Proof is given in [24, 25].

### 3. Convex linear combination of Hurwitz polynomials of positive linear systems

The polynomial

$$p(s) := s^n + \bar{a}_{n-1} s^{n-1} + \dots + \bar{a}_1 s + \bar{a}_0 \quad (7)$$

is called Hurwitz if its zeros  $s_i, i = 1, \dots, n$  satisfy the condition  $\text{Re } s_i < 0$  for  $i = 1, \dots, n$ .

**Definition 3.** The polynomial

$$p(s) := (1 - k)p_1(s) + kp_2(s) \text{ for } k \in [0, 1] \quad (8)$$

is called convex linear combination of the polynomials

$$\begin{aligned} p_1(s) &= s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 \\ p_2(s) &= s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0. \end{aligned} \quad (9)$$

**Theorem 3.** The convex linear combination (8) of the Hurwitz polynomials (9) of the positive linear system is also a Hurwitz polynomial.

**Proof.** By Theorem 2 the polynomials (9) are Hurwitz if and only if

$$a_i > 0 \text{ and } b_i > 0 \text{ for } i = 0, 1, \dots, n - 1. \quad (10)$$

The convex linear combination (8) of the Hurwitz polynomials (9) is a Hurwitz polynomial if and only if

$$\begin{aligned} (1 - k)a_i + kb_i &> 0 \text{ for } k \in [0, 1] \\ \text{and } i &= 0, 1, \dots, n - 1. \end{aligned} \quad (11)$$

Note that the conditions (10) are always satisfied if (11) holds. Therefore, the convex linear combination (8) of the Hurwitz polynomials (9) of the positive linear system is always the Hurwitz polynomial.  $\square$

**Example 1.** Consider the convex linear combination (8) of the Hurwitz polynomials

$$\begin{aligned} p_1(s) &= s^2 + 5s + 2 \\ p_2(s) &= s^2 + 3s + 4. \end{aligned} \quad (12)$$

The convex linear combination (8) of the polynomials (12) is a Hurwitz polynomial since

$$\begin{aligned} (1 - k)5 + 3k &= 5 - 2k > 0 \text{ and} \\ (1 - k)2 + 4k &= 2 + 2k > 0 \text{ for } k \in [0, 1]. \end{aligned} \quad (13)$$

The above considerations for two polynomials (9) of the same order  $n$  can be extended to two polynomials of different orders as follows.

The convex linear combination of two polynomials of different orders

$$\begin{aligned} p_1(s) &= s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 \\ \bar{p}_2(s) &= s^{n-j} + \bar{a}_{n-j-1} s^{n-j-1} + \dots + \bar{a}_1 s + \bar{a}_0. \end{aligned} \quad (14)$$

$j = 0, 1, \dots, n - 1$

is defined by

$$\begin{aligned} \bar{p}(s) &= (1 - k)s^n + (1 - k)a_{n-j+1} s^{n-j+1} + \\ &= (1 - k)a_{n-j} s^{n-j} + ks^{n-j} + \dots + \\ &= (1 - k)a_1 s + k\bar{a}_1 s + (1 - k)a_0 + k\bar{a}_0 \\ &= (1 - k)a_1 s + k\bar{a}_1 s + (1 - k)a_0 + k\bar{a}_0 \end{aligned} \quad (15)$$

for  $0 \leq k < 1$ .

**Theorem 3'.** The convex linear combination (15) of the Hurwitz polynomials (14) of the positive linear system is also a Hurwitz polynomial.

Proof is similar to the proof of Theorem 3.

**Example 2.** Consider the convex linear combination (15) of the Hurwitz polynomials

$$\begin{aligned} p_1(s) &= s^3 + 3s^2 + 2s + 1 \\ p_2(s) &= s^2 + 2s + 3. \end{aligned} \quad (16)$$

The convex linear combination (15) of the Hurwitz polynomials (16)

$$\begin{aligned} \bar{p}(s) &= (1-k)s^3 + (1-k)3s^2 + ks^2 + (1-k)2s + \\ &+ k2s + (1-k) + 3k \text{ for } 0 \leq k < 1 \end{aligned} \quad (17)$$

is also Hurwitz polynomial since its all coefficients are positive (Theorem 2).

#### 4. Extension of Kharitonov theorem to positive interval linear systems

Consider the set of positive interval linear continuous-time systems with the characteristic polynomials

$$p(s) = p_n s^n + p_{n-1} s^{n-1} + \dots + p_1 s + p_0 \quad (18a)$$

where

$$0 < \underline{p}_i \leq p_i \leq \bar{p}_i, \quad i = 0, 1, \dots, n. \quad (18b)$$

**Theorem 4.** The positive interval linear system with the characteristic polynomial (18a) is asymptotically stable if and only if the conditions (18b) are satisfied.

**Proof.** By Kharitonov theorem the set of polynomials (18) is asymptotically stable if and only if the polynomials (6) are asymptotically stable. Note that the coefficients of polynomials (6) are positive if the conditions (18b) are satisfied. Therefore, by Theorem 2 the positive interval linear system with the characteristic polynomials (18a) is asymptotically stable if and only if the conditions (18b) are satisfied.  $\square$

**Example 3.** Consider the positive linear system with the characteristic polynomial

$$p(s) = a_3 s^3 + a_2 s^2 a_1 s + a_0 \quad (19a)$$

with the interval coefficients

$$\begin{aligned} 0.5 \leq a_3 \leq 2, \quad 1 \leq a_2 \leq 3, \\ 0.4 \leq a_1 \leq 1.5, \quad 0.3 \leq a_0 \leq 4. \end{aligned} \quad (19b)$$

By Theorem 4 the interval positive linear system with (19) is asymptotically stable since the coefficients  $a_k$ ,  $k = 0, 1, 2, 3$  of the polynomial (19a) are positive, i.e. the lower and upper bounds are positive.

Consider the interval positive linear continuous-time system

$$\dot{x} = Ax \quad (20)$$

where  $x = x(t) \in \mathfrak{R}^n$  is the state vector and the matrix  $A \in M_n$  is defined by

$$A_1 \leq A \leq A_2 \text{ or equivalently } A \in [A_1, A_2]. \quad (21)$$

**Definition 4.** The interval positive system (20) is called asymptotically stable if the system is asymptotically stable for all matrices  $A \in M_n$  satisfying the condition (21).

By condition (4) of Theorem 2 the positive system (20) is asymptotically stable if there exists strictly positive vector  $\lambda > 0$  such that the condition (4) is satisfied.

For two positive linear systems

$$\dot{x}_1 = A_1 x_1, \quad A_1 \in M_n \quad (22a)$$

and

$$\dot{x}_2 = A_2 x_2, \quad A_2 \in M_n \quad (22b)$$

there exists a strictly positive vector  $\lambda \in \mathfrak{R}_+^n$  such that

$$A_1 \lambda < 0 \text{ and } A_2 \lambda < 0 \quad (23)$$

if and only if the systems (22) are asymptotically stable.

**Example 4.** Consider the positive linear continuous-time systems (22) with the matrices

$$A_1 = \begin{bmatrix} -0.6 & 0.3 \\ 0.4 & -0.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.6 & 0.3 \\ 0.3 & -0.4 \end{bmatrix}. \quad (24)$$

It is easy to verify that for  $\lambda^T = [0.8 \ 1]$  we have

$$\begin{aligned} A_1 \lambda &= \begin{bmatrix} -0.6 & 0.3 \\ 0.4 & -0.4 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.18 \\ -0.18 \end{bmatrix} < 0 \\ A_2 \lambda &= \begin{bmatrix} -0.6 & 0.3 \\ 0.3 & -0.4 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.18 \\ -0.16 \end{bmatrix} < 0. \end{aligned} \quad (25)$$

Therefore, by the condition (4) of Theorem 2 the positive systems are asymptotically stable.

**Theorem 5.** If the matrices  $A_1$  and  $A_2$  of positive systems (22) are asymptotically stable then their convex linear combination

$$A = (1-k)A_1 + kA_2 \text{ for } 0 \leq k < 1 \quad (26)$$

is also asymptotically stable.

**Proof.** By condition (4) of Theorem 2 if the positive linear systems (22) are asymptotically stable then there exists strictly positive vector  $\lambda \in \mathfrak{R}_+^n$  such that

$$A_1\lambda < 0 \text{ and } A_2\lambda < 0. \quad (27)$$

Using (26) and (27) we obtain

$$A\lambda = [(1-k)A_1 + kA_2]\lambda = (1-k)A_1\lambda + kA_2\lambda < 0 \text{ for } 0 \leq k < 1. \quad (28)$$

Therefore, if the positive linear systems (22) are asymptotically stable and (27) hold then their convex linear combination is also asymptotically stable.  $\square$

If  $A_k \in M_n$ ,  $k = 1, 2$  are Hurwitz and  $A_1A_2 = A_2A_1$  then  $\lambda \in \mathfrak{R}_+^n$  satisfying  $A_k\lambda < 0$ ,  $k = 1, 2$  can be chosen in the form  $\lambda = A_1A_2c$ , where  $c \in \mathfrak{R}_+^n$  is strictly positive (see Lemma A1 in Appendix).

**Theorem 6.** The interval positive system (22) are asymptotically stable if and only if the positive linear systems (22) are asymptotically stable.

**Proof.** By condition (4) of Theorem 2 if the matrices  $A_1 \in M_n$ ,  $A_2 \in M_n$  are asymptotically stable then there exists a strictly positive vector  $\lambda \in \mathfrak{R}_+^n$  such that (27) holds. The convex linear combination (26) satisfies the condition  $A\lambda < 0$  if and only if (27) holds. Therefore, the interval system (20) is asymptotically stable if and only if the positive linear system are asymptotically stable.  $\square$

**Example 5.** Consider the interval positive linear continuous-time systems (20) with the matrices

$$A_1 = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}, A_2 = \begin{bmatrix} -3 & 2 \\ 4 & -4 \end{bmatrix}. \quad (29)$$

Using the condition (4) of Theorem 2 are choose for  $A_1$  (given by (29))  $\lambda_1 = [1 \ 1]^T$  and we obtain

$$A_1\lambda_1 = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} < 0, \quad (30a)$$

and for  $A_2$ ,  $\lambda_2 = [0.8 \ 1]^T$

$$A_2\lambda_2 = \begin{bmatrix} -3 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.4 \\ -0.8 \end{bmatrix} < 0, \quad (30b)$$

Therefore, the matrices (29) are Hurwitz. Note that

$$A_1\lambda_2 = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.6 \\ -1.4 \end{bmatrix} < 0, \quad (31)$$

Therefore, for both matrices (29) we may choose  $\lambda = \lambda_1 = \lambda_2 = [0.8 \ 1]^T$  and by Theorem 6 the interval positive system (20) with (29) is asymptotically stable.

## 5. Concluding remarks

The asymptotic stability of interval positive linear continuous-time systems has been investigated. It has been shown that the convex linear combination of the Hurwitz polynomial of positive linear systems is also the Hurwitz polynomial (Theorems 3 and 3'). The Kharitonov theorem has been extended to positive interval linear systems (Theorem 4). The asymptotic stability of interval positive systems described by the state equation (20) has been also analyzed. It has been shown that the interval positive systems (20) is asymptotically stable if and only if the positive systems (22) are asymptotically stable (Theorem 6). The considerations have been illustrated by numerical examples.

The above considerations can be extended to positive linear discrete-time systems and to fractional linear systems. An open problem is an extension of these considerations to standard (nonpositive) linear systems.

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## REFERENCES

- [1] T. Kaczorek, "Positive fractional continuous-time linear systems with singular pencils", *Bull. Pol. Ac.: Tech.* 60 (1), 9–1 (2012).
- [2] T. Kaczorek, *Positive 1D and 2D Systems*, Springer-Verlag, London, 2000.
- [3] T. Kaczorek, "Analysis of positivity and stability of fractional discrete-time nonlinear systems", *Bull. Pol. Ac.: Tech.* 64 (3), 491–494 (2016).
- [4] T. Kaczorek, "Analysis of positivity and stability of discrete-time and continuous-time nonlinear systems", *Computational Problems of Electrical Engineering* 5 (1), (2015).
- [5] T. Kaczorek, "Descriptor positive discrete-time and continuous-time nonlinear systems", *Proc. of SPIE*, vol. 9290 (2014).
- [6] T. Kaczorek, "Positivity and stability of discrete-time nonlinear systems", *IEEE 2nd International Conference on Cybernetics*, 156–159 (2015).
- [7] T. Kaczorek, "Stability of fractional positive nonlinear systems", *Archives of Control Sciences*, 25 (4), 491–496 (2015).
- [8] M. Busłowicz, "Stability analysis of continuous-time linear systems consisting of n subsystems with different fractional orders", *Bull. Pol. Ac.: Tech.* vol. 60, no. 2, 2012, 279–284.
- [9] T. Kaczorek, "Fractional positive continuous-time linear systems and their reachability", *Int. J. Appl. Math. Comput. Sci.* 18 (2), 223–228 (2008).
- [10] T. Kaczorek, "Positive linear systems with different fractional orders", *Bull. Pol. Ac.: Tech.* 58 (3), 453–458 (2010).
- [11] T. Kaczorek, "Positive linear systems consisting of n subsystems with different fractional orders", *IEEE Trans. on Circuits and Systems*, 58 (7), 1203–1210 (2011).

- [12] Ł. Sajewski, “Descriptor fractional discrete-time linear system with two different fractional orders and its solution”, *Bull. Pol. Ac.: Tech.* 64 (1), 15–20 (2016).
- [13] W. Xiang-Jun, W. Zheng-Mao, and L. Jun-Guo, “Stability analysis of a class of nonlinear fractional-order systems”, *IEEE Trans. Circuits and Systems-II, Express Briefs*, 55 (11), 1178–118 (2008).
- [14] T. Kaczorek, “Positive singular discrete-time linear systems”, *Bull. Pol. Ac.: Tech.* 45 (4), 619–631 (1997).
- [15] T. Kaczorek, “Application of Drazin inverse to analysis of descriptor fractional discrete-time linear systems with regular pencils”, *Int. J. Appl. Math. Comput. Sci.* 23 (1), 29–34 (2013).
- [16] H. Zhang, D. Xie, H. Zhang, and G. Wang, “Stability analysis for discrete-time switched systems with unstable subsystems by a mode-dependent average dwell time approach”, *ISA Transactions*, 53, 1081–1086 (2014).
- [17] J. Zhang, Z. Han, H. Wu, and J. Hung, “Robust stabilization of discrete-time positive switched systems with uncertainties and average dwell time switching”, *Circuits Syst. Signal Process.*, 33, 71–95 (2014).
- [18] Ł. Sajewski, “Descriptor fractional discrete-time linear system and its solution – comparison of three different methods”, *Challenges in Automation, Robotics and Measurement Techniques, Advances in Intelligent Systems and Computing*, vol. 440, 2016, 37–50.
- [19] M. Busłowicz, “Stability of linear continuous-time fractional order systems with delays of the retarded type”, *Bull. Pol. Ac.: Tech.* 56 (4), 319–324 (2008).
- [20] M. Busłowicz and T. Kaczorek, “Simple conditions for practical stability of positive fractional discrete-time linear systems”, *Int. J. Appl. Math. Comput. Sci.* 19 (2), 263–169 (2009).
- [21] L. Farina and S. Rinaldi, *Positive Linear Systems; Theory and Applications*, J. Wiley, New York, 2000.
- [22] T. Kaczorek, “Positivity and stability of standard and fractional descriptor continuous-time linear and nonlinear systems”, *Int. J. of Nonlinear Sciences and Num. Simul.* (2017) (in press)
- [23] A. Berman and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, 1994.
- [24] T. Kaczorek, *Theory of Control and Systems*, PWN, Warszawa, 1993 (in Polish).
- [25] V.L. Kharitonov, “Asymptotic stability of an equilibrium position of a family of systems of differential equations”, *Differentsialnye uravneniya*, 14, 2086–2088 (1978).

## 6. Appendix

**Lemma A.1.** If  $A_k \in M_n$ ,  $k = 1, 2$  are Hurwitz and

$$A_1 A_2 = A_2 A_1 \tag{A.1}$$

then the strictly positive vector  $\lambda \in \mathfrak{R}_+^n$  satisfying  $A_k \lambda < 0$ ,  $k = 1, 2$  has the form

$$\lambda = A_1^{-1} A_2^{-1} c, \quad c \in \mathfrak{R}_+^n \text{ strictly positive.} \tag{A.2}$$

**Proof.** Using (A.1) it is easy to prove that

$$A_1^{-1} A_2^{-1} = A_2^{-1} A_1^{-1}. \tag{A.3}$$

It is well-known [11] that if  $A_k \in M_n$ ,  $k = 1, 2$  are Hurwitz, then  $-A_k^{-1} \in \mathfrak{R}_+^{n \times n}$  for  $k = 1, 2$  and  $\lambda$  defined by (A.2) is strictly positive.

Using (A.2) and (A.3) we obtain

$$\begin{aligned} A_1 \lambda &= A_1 A_1^{-1} A_2^{-1} c = -A_2^{-1} c < 0 \quad \text{and} \\ A_2 \lambda &= A_2 A_1^{-1} A_2^{-1} c = A_2 A_2^{-1} A_1^{-1} c = -A_1^{-1} c < 0 \end{aligned} \tag{A.4}$$

$c \in \mathfrak{R}_+^n$  is strictly positive.  $\square$