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Practical and asymptotic stability of fractional discrete-time scalar systems described by a new model

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The stability problems of fractional discrete-time linear scalar systems described by the new model are considered. Using the classical D-partition method, the necessary and sufficient conditions for practical stability and asymptotic stability are given. The considerations are il-lustrated by numerical examples.

Key words: asymptotic stability, practical stability, fractional order, discrete-time linear system.

1. Introduction

The analysis and synthesis of dynamical system described by fractional equations have been recently investigated. A variety of fractional models can be found in various fields (e.g. diffusion, fluid flow, turbulence, viscoelasticity and polymer physics). The fractional calculus and its application has been presented in monographs and papers (see, e.g. [1, 12, 14, 15, 17, 20, 23]).

The main issue in the dynamical systems theory is the stability problem. In the case of linear continuous-time fractional systems this problem has been considered in many publications (see, e.g. [2, 3, 6, 12, 15, 19]). The stability problem of linear discrete-time fractional order systems is more complicated. It results from the fact that the asymptotic stability of such systems corresponds to the asymptotic stability of the associated infinite-dimensional discrete-time systems of natural order with delays. In the practice the number of delays is limited by the length of practical implementation and the asymptotic stability of these system is the so-called practical stability of the fractional discrete-time system. The conditions for practical stability of fractional discrete-time systems has been considered for positive systems [4, 7, 11, 12], non-positive (standard) systems

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[4, 9] and scalar systems with delay [5, 21, 22]. The problem of asymptotic stability of fractional discrete-time systems has been derived in [8, 18, 24].

The aim of this paper is to establish new stability conditions of fractional discrete-time linear scalar systems described by the new model introduced in [13]. The practical stability and asymptotic stability will be analysed. New necessary and sufficient condition for practical stability and asymptotic stability will be proposed.

The fractional models are used to modeling real processes and phenomena. The model considered in this paper can be applied not only for scalar systems, but also for multidimensional systems with diagonal state operator. The example of such a class of real, physical processes are heat transfer processes described by a semigroup model [16].

2. Problem formulation

Two fractional order discrete-time state-space models of linear system have been analyzed in the paper [13]. The new model has been introduced and solution of this model has been presented. The state equations of this model has the form

$$\begin{aligned} &\Delta^{\alpha}x(k) = Ax(k) + Bu(k), \quad k = \{0,1,\ldots\}, \quad \alpha \in (0,1), \\ &y(k) = Cx(k) + Du(k) \end{aligned} \tag{1}$$

with the initial condition $x(0)$, where $x(k) \in \Re^n$, $u(k) \in \Re^m$, $y(k) \in \Re^p$ are the state, input and output vectors, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

The second model in [13] is better known and more analysed in the literature. The state equation of this model has the form

$$\Delta^{\alpha}x(k+1) = Ax(k) + Bu(k), \quad k = \{0,1,\ldots\}, \quad \alpha \in (0,1). \tag{2}$$

The stability conditions of the model (2) was presented for example in [8, 18, 24].

Taking into account the model (1), let us consider the scalar system described by the homogeneous equation

$$\Delta^{\alpha}x(k) = ax(k), \quad k = \{0,1,\ldots\}, \quad \alpha \in (0,1), \tag{3}$$

where a is the scalar.

In this paper the following definition of the fractional difference [12] will be used

$$\Delta^{\alpha}x(k) = \sum_{i=0}^{k} c_i(\alpha)x(k-i) \tag{4}$$

where $\alpha \in \Re$ is the order of the fractional difference and

$$c_i(\alpha) = \begin{cases} 0 & \text{for } i < 0 \\ 1 & \text{for } i = 0 \\ (-1)^i \frac{\alpha(\alpha-1)\ldots(\alpha-i+1)}{i!} & \text{for } i > 0 \end{cases} \tag{5}$$

The coefficients (5) can be calculated using the following formula

$$c_{i+1}(\alpha) = c_i(\alpha)\frac{i-1-\alpha}{i}, \quad i = 1, 2, \ldots \tag{6}$$

with $c_1(\alpha) = -\alpha$.

Using the definition (4) equation (3) can be written in the form

$$x(k) = -[1-a]^{-1}\sum_{i=1}^{k} c_i(\alpha)x(k-i). \tag{7}$$

where $a \neq 1$.

Note that the equation (7) represents a linear discrete-time system with increasing number of delays in state.

From (5) and (6) it follows that the coefficients $c_i(\alpha)$ are negative for $\alpha \in (0, 1)$ and absolute value decrease rapidly to 0 with an increase of i. Therefore, we can assume that the value of i in the equation (7) may be limited by some natural number L. This number is called the length of the practical implementation. Thus, the equation (7) has the form

$$x(k) = -[1-a]^{-1}\sum_{i=1}^{L} c_i(\alpha)x(k-i). \tag{8}$$

The equation (8) represents a linear discrete-time system with L delays in state. Moreover, the system (8) is called the practical realization of fractional system (3).

The definition of practical stability and the related definition of asymptotic stability for fractional discrete-time systems have been introduced in the work [11]. With regard to the system (3) these definitions take the following forms.

Definition 1. *The fractional system* (3) *is called practically stable if the system* (8) *is asymptotically stable.*

Definition 2. *The fractional system* (3) *is called asymptotically stable if the system* (8) *is practically stable for* $L \to \infty$.

Using the stability theory of discrete-time linear systems and Definition 1 we have the following theorem.

Theorem 1. *The fractional system* (3) *with given length* L *of practical implementation is practically stable if and only if*

$$w(z) \neq 0, \quad |z| \geqslant 1, \tag{9}$$

where

$$w(z) = 1 + [1-a]^{-1}\sum_{i=1}^{L} c_i(\alpha)z^{-i} \tag{10}$$

is the characteristic polynomial of the system (8).

The characteristic equation $w(z) = 0$ of the system (8) can be written as

$$z^L + [1-a]^{-1}\sum_{i=1}^{L} c_i(\alpha)\, z^{L-i} = 0. \tag{11}$$

To analyse practical stability of the fractional system (3) we can used well-known methods for testing the asymptotic stability of discrete-time systems (8). However, this is not an easy task in the case of the high degree of the equation (11) which depends on the length L of practical implementation.

The main aim of this paper is to give new necessary and sufficient conditions for practical stability and asymptotic stability of the system (3), which do not require direct checking of the condition (9). Proposed stability conditions do not require a priori knowledge of the characteristic polynomial (10).

3. Solution of the problem

For the stability analysis of fractional discrete-time system (3), without reducing generality of considerations, we consider the system described by the equation

$$\Delta^{\alpha} x(k) = (a+jb)\, x(k), \quad j^2 = -1, \quad \alpha \in (0,\, 1), \tag{12}$$

where a and b are real numbers.

For the system (12) equation (8) takes the form

$$x(k) = -[1-(a+jb)]^{-1}\sum_{i=1}^{L} c_i(\alpha)\, x(k-i). \tag{13}$$

The characteristic polynomial of the system (13) is the polynomial with complex coefficients of the form

$$\widetilde{w}(z) = 1 + [1-a-jb]^{-1}\sum_{i=1}^{L} c_i(\alpha)\, z^{-i}. \tag{14}$$

The D-decomposition method [10] will be applied to the stability analysis of the system (13) in connection with values of the parameters a and b. Using this method the stability region in the parameter plane $(a,\, b)$ may be determined and the parameters can be specified. The plane $(a,\, b)$ is decomposed by the boundaries of D-decomposition into finite number regions $D(q)$. The polynomial (14) for any point in the region $D(q)$ has q zeros which satisfy the condition $|z| > 1$. The stability region of polynomial (14) is the region denoted as D(0). For any point in the D-decomposition boundaries, the polynomial (14) has at least one zero on the unit circle in the complex z-plane. These zeros may be real or complex, thus, we have the real zero boundary and the complex zero boundary. Any point in the real zero boundary corresponds to such values of a and b for which the polynomial (14) has zeros $z = 1$ or $z = -1$. While any point in the complex zero boundary corresponds to such values of a and b for which the polynomial (14) has complex zeros satisfying the condition $|z| = 1$.

Firstly, the real zero boundary will be calculated. For $z = 1$ and $z = -1$ from the equation $\widetilde{w}(z) = 0$ after transformation we obtain, respectively,

$$a + jb = 1 + \sum_{i=1}^{L} c_i(\alpha), \tag{15}$$

$$a + jb = 1 + \sum_{i=1}^{L} c_i(\alpha)(-1)^{-i}. \tag{16}$$

Hence, in the plane (a, b) the real zero boundaries are two points: point with coordinates (corresponding to $z = 1$)

$$1 + \sum_{i=1}^{L} c_i(\alpha), \quad b = 0, \tag{17}$$

and point with coordinates (corresponding to $z = -1$)

$$1 + \sum_{i=1}^{L} c_i(\alpha)(-1)^{-i}, \quad b = 0. \tag{18}$$

Now, the complex zero boundary will be obtained. This boundary is determined by solving, with respect to a and b, the complex equation

$$\widetilde{w}(z) = 1 + [1 - a - jb]^{-1} \sum_{i=1}^{L} c_i(\alpha) \exp(-j\omega i). \tag{19}$$

This equation is obtained by substitution $z = \exp(j\omega)$, $\omega \in [0, 2\pi]$ (boundary of the unit circle in the complex z-plane) in the polynomial (14) and equating to 0. Finally, by solving the equations (19) we get

$$a(\omega) = 1 + \sum_{i=1}^{L} c_i(\alpha) \cos(\omega i), \tag{20}$$

$$b(\omega) = -\sum_{i=1}^{L} c_i(\alpha) \sin(\omega i). \tag{21}$$

Equations (20) and (21) determine the complex zero boundary in plane (a, b). Note that from these equations for $\omega = 0$ and $\omega = \pi$ we obtain formulas (17) and (18), respectively.

The practical stability regions of the system (12), that is the asymptotic stability regions of the system (13), for given $L = 5$ and $\alpha = 0.5$ is shown in Fig. 1. The complex zero boundary obtained for $\omega \in [0, 2\pi]$ divides the plane (a, b) into bounded and unbounded regions. The real zero boundary is denoted by x-marks in Fig. 1. The asymptotic stability region D(0) of system (13) is chosen by testing an arbitrary point from each region and checking the asymptotic stability of the polynomial (14). For example, choosing the point with coordinates $a = 0.8$ and $b = 0$ we obtain the following zeros of polynomial (14): $z_1 = 2.777$, $z_{2,3} = -0.353 \pm j0.275$, $z_{4,5} = 0.214 \pm j0.448$. For the zero z_1 the condition $|z| < 1$ does not hold, thus, the region with this point is the unstable region. Choosing the point with coordinates $a = 0$ and $b = 0$ we obtain the following zeros of polynomial (14): $z_1 = 0.85$, $z_{2,3} = -0.322 \pm j0.235$, $z_{4,5} = 0.147 \pm j0.426$. For all these zeros the condition $|z| < 1$ holds, thus, the regions with this point are the stability regions D(0). Hence, in the plane (a, b) the practical stability region of the system (12) is the unbounded region. The bounded region by the closed curve $a(\omega) + jb(\omega)$, where $a(\omega)$ and $b(\omega)$ are calculated from (20) and (21) is the unstable region.

Fig. 2 shows the practical stability boundaries of the system (12) with $\alpha = 0.5$ and different values of L. The practical stability boundaries with $L = 1000$ and different values of α are shown in Fig. 3. It is easy to check that for $\alpha = 1$ the stability boundary is the circle with centre at point (1,0) and radius 1.

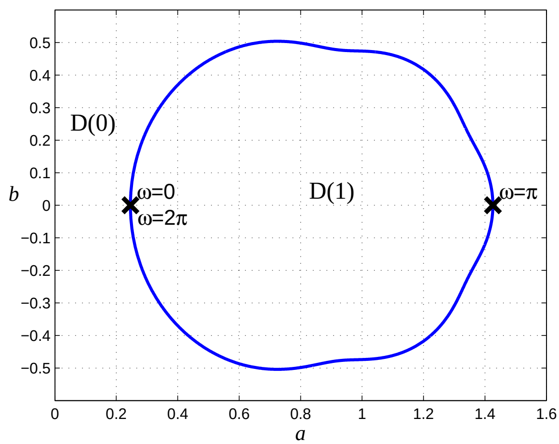

Figure 1: The practical stability region D(0) of the system (12) for $\alpha = 0.5$, $L = 5$.

The state equation of system (3) may be obtained by assumption $b = 0$ in (12). Therefore, for the fractional system (3) the practical stability region D(0) shown in Fig. 1 reduces to the intervals of real axis. The endpoints of this intervals correspond to the real zero boundaries (17) and (18). These endpoints can be obtained directly from (15) and

(16) for $b = 0$ and denoted by $d(\alpha,L)$ and $g(\alpha,L)$ accordingly. Then we have

$$d(\alpha,L) = 1 + \sum_{i=1}^{L} c_i(\alpha), \tag{22}$$

$$g(\alpha,L) = 1 + \sum_{i=1}^{L} c_i(\alpha)(-1)^{-i}. \tag{23}$$

The values of $d(\alpha,L)$ and $g(\alpha,L)$ depend on the given order $\alpha \in (0, 1)$ and the given length L of practical implementation.

From the above we have the following theorem.

Theorem 2. *The fractional system* (3) *with the given length L of practical implementation is practically stable if and only if*

$$a < d(\alpha,L) \quad or \quad a > g(\alpha,L), \tag{24}$$

where $d(\alpha,L)$ *and* $g(\alpha,L)$ *are computed from the formulas* (22) *and* (23).

The diagrams of $d(\alpha,L)$ and $g(\alpha,L)$ calculated from (22) and (23) as a function of fractional order $\alpha \in (0, 1)$ for small $L = 10$ and large $L = 1000$ values of the length L of practical implementation are shown in Fig. 4. The practical stability regions for a given L determines values of a for which the system (3) is practically stable with a given $\alpha \in (0, 1)$. Fig. 4 shows that, for fixed $\alpha \in (0, 1)$, values of $d(\alpha,L)$ differ significantly for small and large values of L, whereas values of $g(\alpha,L)$ differ slightly for small and large values of L.

Example 1. Consider the fractional system (3) with $\alpha = 0.1$. Find values of coefficient a for which the system is practically stable for $L = 10$ and $L = 1000$.

Using Theorem 2 and Fig. 4 we obtain that the system (3) with $\alpha = 0.1$ is practically stable for $a < 0.469$ and $a > 1.072$ if $L = 10$ and for $a < 0.74$ and $a > 1.068$ if $L = 1000$. For example, the system with $a = 0.6$ is practically stable for $L = 10$ but it is not practically stable for $L = 1000$.

Now we consider the problem of asymptotic stability. The fractional system (3) is asymptotically stable if the system (8) is practically stable for $L \to \infty$.

Using the formula [12]

$$\sum_{i=1}^{L} c_i(\alpha) = -1, \quad \alpha \in (0, 1), \tag{25}$$

from (22) for $L \to \infty$ we get

$$\lim_{L \to \infty} d(\alpha,L) = 0. \tag{26}$$

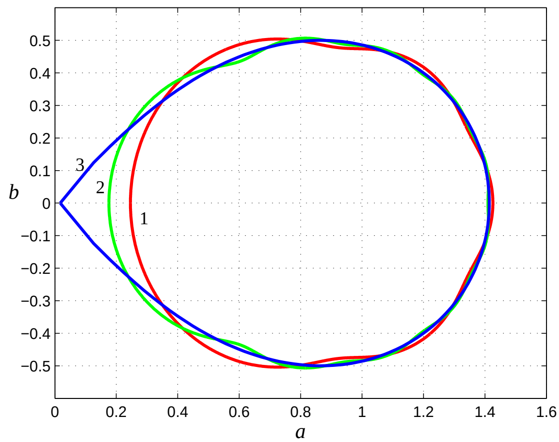

Figure 2: The boundaries of the practical stability regions of system (12) for $\alpha = 0.5$ and $L = 5$ (boundary 1), $L = 10$ (boundary 2), $L = 1000$ (boundary 3).

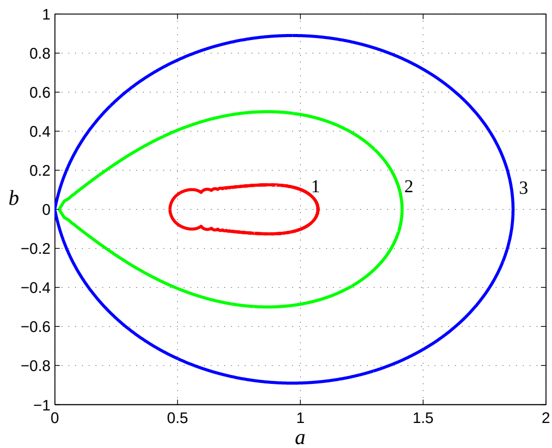

Figure 3: The boundaries of the practical stability regions of system (12) for $L = 1000$ and $\alpha = 0.1$ (boundary 1), $\alpha = 0.5$ (boundary 2), $\alpha = 0.9$ (boundary 3).

We consider the following equality (for $\alpha > 0$ and $|y| \leqslant 1$)

$$(1+y)^{\alpha} = 1 + \alpha y + \frac{\alpha(\alpha-1)}{2!}y^2 + \dots + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-i+1)}{i!}y^i + \cdots$$

For $y = 1$ we have

$$2^{\alpha} = 1 + \alpha + \frac{\alpha(\alpha-1)}{2!} + \dots + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-i+1)}{i!} + \cdots \tag{27}$$

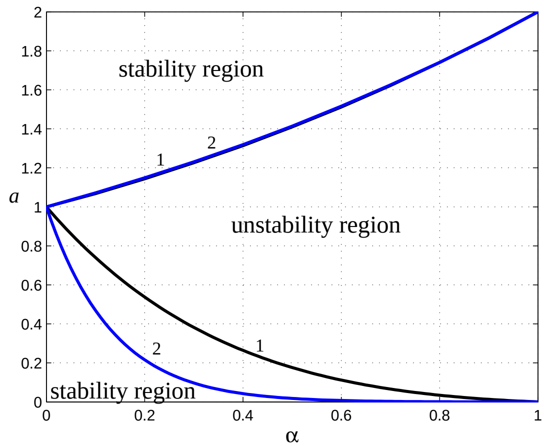

Figure 4: The practical stability regions of system (3) in the parameter plane (α, a) for $L = 10$ (boundaries 1) and $L = 1000$ (boundaries 2).

From (5) and (27) it follows that

$$2^{\alpha} = 1 + \sum_{i=1}^{\infty} (-1)^i c_i(\alpha). \tag{28}$$

Hence, for $L \to \infty$ from (23) we have

$$\lim_{L\to\infty} g(\alpha, L) = 2^{\alpha}. \tag{29}$$

Equations (26) and (29) are the real zero boundaries for $L \to \infty$. Taking into account above and Theorem 2 we obtain the following necessary and sufficient condition for asymptotic stability of the fractional discrete-time linear scalar system (3).

Theorem 3. *The fractional system* (3) *is asymptotically stable if and only if*

$$a < 0 \quad or \quad a > 2^{\alpha}. \tag{30}$$

Example 2. Consider the fractional system (3) with $\alpha = 0.5$. Check the asymptotic stability of this system.

According to Theorem 3 this system is asymptotically stable when

$$a < 0 \quad or \quad a > 1.414. \tag{31}$$

Taking into considerations Theorem 3 and relationship $2^{\alpha} \in (1, 2)$ for all $\alpha \in (0, 1)$ we have the following lemma.

Lemma 1. *If*

1. $a < 0$ *or* $a > 2$ *then the fractional system* (3) *is asymptotically stable for any* $\alpha \in (0, 1)$,
2. $1 < a < 2$ *then the fractional system* (3) *is asymptotically stable for*

$$0 < \alpha < \log_2 a \tag{32}$$

where $\log_2 a$ *is the base 2 logarithm of a.*

Example 3. Consider the fractional system (3) with $a = 1.6$. Find values of fractional order α for which the system is asymptotically stable.

From Lemma 1 we have $0 < \alpha < \log_2 0.6 = 0.678$. Hence, the system is asymptotically stable if and only if $\alpha \in (0, 0.678)$.

4. Concluding remarks

The practical stability and asymptotical stability problem of discrete-time linear scalar system (3) of fractional order $\alpha \in (0, 1)$ is analysed. Using the classical D-partition method new necessary and sufficient condition for practical stability (Theorem 2) and new necessary and sufficient condition for asymptotic stability (Theorem 3) are established.

The work can be extended for a class of systems described by the equation $\Delta^{\alpha} x(k) = Ax(k)$, with diagonal state space matrix A.

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