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A new method for determination of positive realizations of linear continuous-time systems

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Abstract. A new method for determination of positive realizations of given transfer matrices of linear continuous-time linear systems is proposed. Necessary and sufficient conditions for the existence of positive realizations of transfer matrices are presented. A procedure for computation of the positive realizations is proposed and illustrated by an example.

Key words: determination, positive, realization, transfer matrix, linear, continuous-time, system.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in the monographs [1, 2]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc. [1, 2].

The determination of the matrices A, B, C, D of the state equations of linear systems for given transfer matrices is called the realization problem. The realization problem is a classical problem of analysis of linear systems, which has been considered in many books and papers [3–7]. A tutorial on the positive realization problem has been given in paper [8] and in books [1, 2]. The minimal realization problem has been analyzed in [9, 10] and the positive minimal realization problem in [11–13]. The realization problem for linear systems with delays has been analyzed in [2, 14–17] and the positive stable realizations in [18–21]. For fractional linear systems the realization problem has been considered in [5, 6, 13, 22–24]. Realization of singular systems via Markov parameters has been introduced in [25] and Digraphs minimal realizations of state matrices for fractional positive systems in [26].

In this paper, a new method for determination of positive realizations of linear continuous-time systems is proposed.

The paper is organized as follows. In Section 2 some definitions and theorems concerning the positive continuous-time linear systems are recalled. A new method for determination of positive realizations for single-input single-output linear systems is proposed in Section 3 and for multi-input multi-output systems in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Preliminaries

Consider the continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Definition 1. [1, 2] System (1) is called (internally) positive if $x(t)\in\Re_+^n$ and $y(t)\in\Re_+^p$, $t\geq 0$ for any initial conditions $x(0)\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\geq 0$.

Theorem 1. [1, 2] System (1) is positive if and only if

$$A\in M_n,\ B\in\Re_+^{n\times m},\ C\in\Re_+^{p\times n},\ D\in\Re_+^{p\times m}. \tag{2}$$

The transfer matrix of the system (1) is given by

$$T(s) = C[I_n s - A]^{-1}B + D. \tag{3}$$

The transfer matrix is called proper if

$$\lim_{s\to\infty} T(s) = D\in\Re_+^{p\times m} \tag{4}$$

and it is called strictly proper if $D = 0$.

Definition 2. [6, 7] Matrices (2) are called a positive realization of $T(s)$ if they satisfy equality (3).

Definition 3. [6, 7] Matrices (2) are called asymptotically stable if matrix A is an asymptotically stable Metzler matrix (Hurwitz Metzler matrix).

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Theorem 2. [6, 7] Positive realization (2) is asymptotically stable if and only if all coefficients of the polynomial

$$p_A(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \quad (5)$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

The positive realization problem can be stated as follows. Given a proper transfer matrix $T(s)$ find its positive realization (2).

Theorem 3. [6] If (2) is a positive realization of (3) then the matrices

$$\bar{A} = PAP^{-1},\ \bar{B} = PB,\ \bar{C} = CP^{-1},\ \bar{D} = D \quad (6)$$

are also a positive realization of (3) if and only if the matrix $P \in \Re_+^{n\times n}$ is a monomial matrix (in each row and in each column only one entry is positive and the remaining entries are zero).

3. Positive realizations of transfer functions

In this section necessary and sufficient conditions will be given for the existence of positive realizations (A, B, C, D) of the given transfer function

$$T(s) = \frac{m_n s^n + m_{n-1}s^{n-1} + \ldots + m_1 s + m_0}{s^n + d_{n-1}s^{n-1} + \ldots + d_1 s + d_0}. \quad (7)$$

Using (4) we obtain

$$D = \lim_{s\to\infty} T(s) = m_n \quad (8)$$

and

$$\bar{T}(s) = T(s) - D \frac{\bar{m}_{n-1}s^{n-1} + \ldots + \bar{m}_1 s + \bar{m}_0}{s^n + d_{n-1}s^{n-1} + \ldots + d_1 s + d_0} = C[I_n s - A]^{-1}B \quad (9a)$$

where

$$\bar{m}_k = m_k - m_n d_k \ \text{ for } \ k = 0, 1, \ldots, n-1. \quad (9b)$$

Theorem 4. There exists the positive realization

$$A = \begin{bmatrix} -s_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -s_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -s_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 1 & -s_n \end{bmatrix},\ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad (10)$$

$$C = [0 \ \cdots \ 0 \ 1],\ D = m_n$$

of transfer function (7) if and only if the following conditions are satisfied:

$$1) \quad m_n \ge 0 \quad (11a)$$

$$2) \quad S^{-1}M \in \Re_+^n, \quad (11b)$$

where s_k, $k = 1, \ldots, n$ are the zeros of the denominator

$$d(s) = s^n + d_{n-1}s^{n-1} + \ldots + d_1 s + d_0 = (s+s_1)(s+s_2)\ldots(s+s_n) \quad (11c)$$

and

$$S = \begin{bmatrix} 1 & s_1 & s_1 s_2 & \cdots & s_1 s_2 \ldots s_{n-1} \\ 0 & 1 & s_1 + s_2 & \cdots & s_1 + s_2 + \ldots + s_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$
$$M = \begin{bmatrix} \bar{m}_0 \\ \bar{m}_1 \\ \vdots \\ \bar{m}_{n-1} \end{bmatrix},\ \bar{m}_0 > 0. \quad (11d)$$

Proof. It is easy to check that

$$C[I_n s - A]^{-1} = C \begin{bmatrix} s+s_1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & s+s_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s+s_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & -1 & s+s_n \end{bmatrix} = \frac{C[I_n s - A]_{ad}}{d(s)} \quad (12a)$$

where product of the matrix C and the adjoint matrix $[I_n s - A]_{ad}$ has the form

$$C[I_n s - A]_{ad} = [1 \ \ s+s_1 \ \ (s+s_1)(s+s_2) \ \cdots \ \cdots \ (s+s_1)(s+s_2)\ldots(s+s_{n-1})] \quad (12b)$$

Using (12) and (10) we obtain

$$C[I_n s - A]^{-1}B = \frac{C[I_n s - A]_{ad}B}{d(s)} = \frac{1}{d(s)}\begin{bmatrix}1 & s+s_1 & (s+s_1)(s+s_2) & \cdots & (s+s_1)(s+s_2)\dots(s+s_{n-1})\end{bmatrix}\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix} =$$

$$= \frac{1}{d(s)}\left[b_1 + b_2(s+s_1) + b_3(s+s_1)(s+s_2) + \ldots + b_n(s+s_1)(s+s_2)\ldots(s+s_{n-1})\right] \text{k} = \tag{13}$$

$$= \frac{b_1 + b_2 s_1 + b_3 s_1 s_2 + \ldots + b_n s_1 s_2 \ldots s_{n-1} + \left[b_2 + b_3(s_1+s_2) + \ldots + b_n(s_1+s_2+\ldots+s_{n-1})\right]s + \ldots + b_n s^{n-1}}{d(s)} =$$

$$= \frac{\bar{m}_{n-1}s^{n-1} + \ldots + \bar{m}_1 s + \bar{m}_0}{d(s)} = \bar{T}(s).$$

and the matrices S, B and M are related by the equation

$$SB = M. \tag{14}$$

The matrix $B \in \Re_+^n$ if and only if the condition (11b) is satisfied.

Note that the realization (10) is positive if and only if the conditions (11a) and (11b) are satisfied. □

Remark 1. The positive realization (10) of (7) is asymptotically stable if and only if $d_k > 0$ for $k = 0, 1, \ldots, n-1$.

Proof. By Theorem 2 the zeros s_k of the polynomial $d(s)$ satisfy the condition $\operatorname{Re} s_k < 0$ for $k = 1, \ldots, n$ if and only if $d_k > 0$ for $k = 0, 1, \ldots, n-1$. □

Example 1. Find the positive realization (10) of the transfer function

$$T(s) = \frac{m_3 s^3 + m_2 s^2 + m_1 s + m_0}{s^3 + d_2 s^2 + d_1 s + d_0} = \frac{2s^3 + 15s^2 + 32s + 24}{s^3 + 6s^2 + 11s + 6}. \tag{15}$$

Using (14), (8) and (9a) we obtain

$$D = \lim_{s\to\infty} T(s) = \lim_{s\to\infty} \frac{2s^3 + 15s^2 + 32s + 24}{s^3 + 6s^2 + 11s + 6} = 2 \tag{16}$$

and

$$\bar{T}(s) = T(s) - D = \frac{3s^2 + 10s + 12}{s^3 + 6s^2 + 11s + 6} = \frac{\bar{m}_2 s^2 + \bar{m}_1 s + \bar{m}_0}{s^3 + d_2 s^2 + d_1 s + d_0}, \tag{17}$$

where $\bar{m}_2 = m_2 - m_3 d_2 = 3$, $\bar{m}_1 = m_1 - m_3 d_1 = 10$, $\bar{m}_0 = m_0 - m_3 d_0 = 12$.

The polynomial

$$d(s) = s^3 + 6s^2 + 11s + 6 = (s+1)(s+2)(s+3) \tag{18}$$

has the zeros: $s_1 = -1$, $s_2 = -2$, $s_3 = -3$ and the matrix A is Hurwitz and has the form

$$A = \begin{bmatrix}-s_1 & 0 & 0\\ 1 & -s_2 & 0\\ 0 & 1 & -s_3\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 1 & -2 & 0\\ 0 & 1 & -3\end{bmatrix}. \tag{19}$$

Using (11d) and (17) we obtain

$$B = \begin{bmatrix}1 & s_1 & s_1 s_2\\ 0 & 1 & s_1+s_2\\ 0 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}\bar{m}_0\\ \bar{m}_1\\ \bar{m}_2\end{bmatrix} = \begin{bmatrix}1 & 1 & 2\\ 0 & 1 & 3\\ 0 & 0 & 0\end{bmatrix}^{-1}\begin{bmatrix}12\\ 10\\ 3\end{bmatrix} = \begin{bmatrix}5\\ 1\\ 3\end{bmatrix} \tag{20}$$

and $C = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$.

The positive asymptotically stable realization of the transfer function (15) is given by (19), (20) and (16).

Remark 2. For the transfer function

$$T(s) = \frac{m_0}{s^n + d_{n-1}s^{n-1} + \ldots + d_1 s + d_0}, \quad m_0 > 0 \tag{21}$$

there always exists the positive realization

$$A = \begin{bmatrix} -s_1 & 0 & 0 & \cdots & 0 & -s_2 \\ 1 & -s_2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -s_n \end{bmatrix}, \quad B = \begin{bmatrix} m_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{22}$$

$$C = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}, \quad D = 0.$$

where s_k, $k = 1, \ldots, n$ are the zeros of (21).

The positive realization is asymptotically stable if and only if $d_k > 0$ for $k = 0, 1, \ldots, n-1$.

Theorem 5. There exists the positive realization

$$\bar{A} = \begin{bmatrix} -s_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -s_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -s_{n-1} & 1 \\ 0 & 0 & 0 & \cdots & 1 & -s_n \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{23}$$

$$\bar{C} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}, \quad D = m_n$$

of the transfer function (7) if and only if the conditions (11a) and (11b) are satisfied, where s_k, $k = 1, \ldots, n$ are the zeros of (11c), S and M are defined by (11d).

Proof. The proof is similar (dual) to the proof of Theorem 4.

Example 2. (Continuation of Example 1) Find the positive realization (23) of the transfer function (14) using Theorem 5.

The matrix D and the transfer function (16) we compute in the same way as in Example 1. Using the zeros $s_1 = -1$, $s_2 = -2$, $s_3 = -3$ of the polynomial (18) we obtain the matrix

$$\bar{A} = \begin{bmatrix} -s_1 & 1 & 0 \\ 0 & -s_2 & 1 \\ 0 & 0 & -s_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}. \tag{24}$$

In this case we have

$$\begin{bmatrix} 1 & s_1 & s_1 s_2 \\ 0 & 1 & s_1 + s_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \bar{m}_0 \\ \bar{m}_1 \\ \bar{m}_2 \end{bmatrix} \tag{25}$$

and

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & s_1 & s_1 s_2 \\ 0 & 1 & s_1 + s_2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{m}_0 \\ \bar{m}_1 \\ \bar{m}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 10 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}. \tag{26}$$

Therefore, the matrices B and C have the forms

$$\bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \bar{C} = \begin{bmatrix} c_1 & c_2 & \cdots & c_3 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \end{bmatrix}. \tag{27}$$

The positive asymptotically stable realization of the transfer function (15) is also given by (24), (27) and (16).

4. Positive realizations for multi-input multi-output systems

In this section the method presented in Section 3 will be extended to multi-input multi-output (MIMO) linear systems. To avoid the loss of generality and to simplify the notation, two-input two-output systems will be considered.

The problem under the considerations can be stated as follows. For given proper transfer matrix

$$T(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix},$$
$$T_{ik}(s) = \frac{m_{ikn} s^n + \ldots + m_{ik1} s + m_{ik0}}{s^n + d_{ikn-1} s^{n-1} + \ldots + d_{ik1} s + d_{ik0}}, \quad i, k = 1, 2 \tag{28}$$

find the positive realization (A, B, C, D) such that

$$T(s) = C[I_n s - A]^{-1} B + D. \tag{29}$$

Using

$$D = \lim_{s \to \infty} T(s) \tag{30}$$

we may find the matrix D and the strictly proper transfer matrix

$$\bar{T}(s) = T(s) - D = C[I_n s - A]^{-1} B = \begin{bmatrix} \frac{\bar{m}_{11}(s)}{d_1(s)} & \frac{\bar{m}_{12}(s)}{d_1(s)} \\ \frac{\bar{m}_{21}(s)}{d_2(s)} & \frac{\bar{m}_{22}(s)}{d_2(s)} \end{bmatrix}, \tag{31}$$

where

$$d_i(s) = s^n + d_{in-1}s^{n-1} + \ldots + d_{i1}s + d_{i0}, \quad i = 1, 2 \quad (32a)$$

is the common least denominator of $T_{i1}(s)$ for $i = 1, 2$ and $s_{i1}, s_{i2}, \ldots, s_{in}$, $i = 1, 2$ are its zeros, i.e.

$$d_i(s) = (s + s_{i1})(s + s_{i2}) \ldots (s + s_{in}), \quad i = 1, 2 \quad (32b)$$

and

$$\bar{m}_{ik}(s) = \bar{m}_{ikn-1}s^{n-1} + \ldots + \bar{m}_{ik1}s + \bar{m}_{ik1} + \bar{m}_{ik0}, \quad i = 1, 2 \quad (33)$$

The matrices A_i of the positive realizations have the form

$$A_i = \begin{bmatrix} -s_{i1} & 0 & 0 & \cdots & 0 & 0 \\ 1 & -s_{i2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -s_{in-1} & 0 \\ 0 & 0 & 0 & \cdots & 1 & -s_{in} \end{bmatrix}, \quad i = 1, 2 \quad (34)$$

and

$$A = \text{blockdiag}\begin{bmatrix} A_1 & A_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}. \quad (35)$$

The matrices B and C have the forms

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad B_{ik} = \begin{bmatrix} b_{ik1} \\ b_{ik2} \\ \\ b_{ikn_i-1} \end{bmatrix} \in \Re_+^{n_i}, \quad i, k = 1, 2 \quad (36)$$

and

$$C = \text{blockdiag}\begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad C_i = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \in \Re_+^{1\times n_i}, \quad i = 1, 2. \quad (37)$$

The entries of B_{ik}, $i, k = 1, 2$ are calculated in the same way as the entries of B in Section 3 using the equation (36).

Therefore, we have the following theorem.

Theorem 6. There exists the positive realization given by (30), (35), (36) and (37) of the transfer matrix (28) if and only if the following conditions are satisfied:

1) $D \in \Re_+^{2\times 2}$ (defined by (30)) (38)

2) $S_i^{-1}M_i \in \Re_+^{n_i}$, $i = 1, 2$, (39a)

where

$$S_i = \begin{bmatrix} 1 & s_{i1} & s_{i1}s_{i2} & \cdots & s_{i1}s_{i2}\ldots s_{in-1} \\ 0 & 1 & s_{i1}+s_{i2} & \cdots & s_{i1}+s_{i2}+\ldots+s_{in-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad (39b)$$

$i = 1, 2$,

$$M_{ik} = \begin{bmatrix} \bar{m}_{ik0} \\ \bar{m}_{ik1} \\ \vdots \\ \bar{m}_{ikn_i-1} \end{bmatrix}, \quad i, k = 1, 2. \quad (39c)$$

Proof. The realization is positive if and only if the condition (38) is satisfied. The matrix A defined by (35) and (34) is a Metzler matrix and it is Hurwitz (asymptotically stable) if $\text{Re}\,s_k < 0$ for $i = 1, 2$ and $k = 1, \ldots, n$. The matrix $B \in \Re_+^{(n_1+n_2)\times 2}$ if and only if the conditions (39) are satisfied. The matrix C defined by (37) is always nonnegative. Therefore, the realization given by (30), (35), (36) and (37) is positive if and only if the conditions (38) and (39) are satisfied. □

From the above considerations we have the following procedure for computation of the positive realization (A, B, C, D) for given transfer matrix (28).

Procedure 1.

Step 1. Knowing $T(s)$ and using (30) and (31) compute the matrix D and the strictly proper transfer matrix $\bar{T}(s)$.

Step 2. Compute the zeros s_{ij}, $i = 1, 2, j = 1, \ldots, n$ of polynomial (33) and matrices (34) and (35).

Step 3. Using (39b) and (39c) compute the matrices S_i and M_{ik}, $i, k = 1, 2$ and check the conditions (39a). If the conditions (39a) are satisfied then there exists $B \in \Re_+^{(n_1+n_2)\times 2}$ and the positive realization of the matrix (28).

Step 4. The desired positive realization is given by (35), (36), (37) and (38).

Example 3. Find the positive realization of the transfer matrix

$$T(s) = \begin{bmatrix} \dfrac{s^2+5s+5}{s^2+3s+2} \\ \dfrac{2s+7}{s+3} \end{bmatrix}. \quad (40)$$

Using Procedure 1 we obtain the following:

Step 1. Using (30), (31) and (40) we obtain

$$D = \lim_{s\to\infty} T(s) = \lim_{s\to\infty} \begin{bmatrix} \dfrac{s^2+5s+5}{s^2+3s+2} \\ \dfrac{2s+7}{s+3} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \quad (41)$$

and

$$\overline{T}(s) = T(s) - D = \begin{bmatrix} \frac{2s+3}{s^2+3s+2} \\ \frac{1}{s+3} \end{bmatrix}. \tag{42}$$

Step 2. The zeros of the polynomial

$$d_1(s) = s^2 + 3s + 2 \tag{43}$$

are $s_{11} = -1$, $s_{12} = -2$ and the polynomial

$$d_2(s) = s + 3 \tag{44}$$

has only one zero $s_{21} = -3$.
In this case the matrix (35) has the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix}. \tag{45}$$

Step 3. Taking into account that in this case

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_{13} \end{bmatrix} \tag{46}$$

and using equation (14) we obtain

$$B_1 = \begin{bmatrix} 1 & -s_{11} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \overline{m}_{10} \\ \overline{m}_{11} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{47a}$$

and

$$B_2 = \begin{bmatrix} b_{13} \end{bmatrix} = 1. \tag{47b}$$

Therefore, matrix B has the form

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \tag{48}$$

and the matrix

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{49}$$

Step 4. The desired positive realization of (40) is given by (45), (48), (49) and (41).

It is easy to check that the matrices

$$\overline{A} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{50}$$

are also the (dual) positive realization of transfer matrix (40).

Remark 3. To the presented method the dual method based on the common denominator for each column of $T(s)$ can be applied.

Remark 4. By Theorem 3 if the matrices A, B, C, D are a positive realization of $T(s)$ then the matrices PAP^{-1}, PB, CP^{-1}, D are also its positive realization for any monomial matrix P.

5. Concluding remarks

A new method for determination of positive realizations of transfer matrices of linear continuous-time systems has been proposed. Necessary and sufficient conditions for the existence of the positive realizations have been established (Theorems 4, 5 and 6). A procedure for computation of the positive realizations has been proposed and illustrated by an example (Example 3). The presented method can be extended to linear discrete-time systems and to linear fractional systems.

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