BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 66, No. 5, 2018
DOI: 10.24425/124281

Optimum tasks and solutions for energy transmission from the source to the receiver

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Abstract. The article presents the minimum optimization tasks that streamline the operating conditions of real linear power sources. Square functions, depending on source current and source power losses within the common power balance condition, have been proposed as quality criteria. These problems have been solved using the exact and simplified approximate methods.

Key words: energy sources, minimum principle, power quality, optimum task.

1. Introduction

The first publications that mention the optimal current solution were released in the 1930s. Fryze calculated the optimal current that delivers given power with the minimum RMS value from the ideal voltage source. At the time, the solution would not have been called optimal as the name was first used in the 1980s. M. Brodzki, M. Pasko, M. Siwczyński and J. Walczak have discussed optimization theory in the following publications: [4, 10, 11, 14, 24]. Further work can be found in the literature [15–22] and in many other items, however this paper is not intended to list out existing solutions. The purpose of this publication is to introduce new solutions, hitherto unknown, and compare them with previously known solutions by other authors. Research based on those (i.e. the new solutions) will appear in future publications.

Generally speaking, the optimizing problem of the source operation involves calculation of the current of a real voltage source in such way that under specified power (or energy) balance conditions the current functional would be minimized.

2. Minimum tasks for the real power source and the relations between them

Figure 1 shows a diagram of the real power source for which three optimization tasks will be formulated: the task of the maximum active power delivered from the source, the task of the minimum current norm and minimum power loss of the source under a pre-set power balance condition.

The voltage-current source relation with the use of the convolution internal impedance operator is written down briefly as:

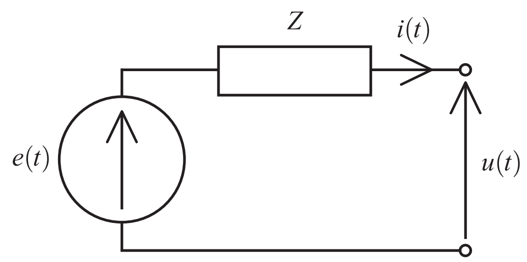

Fig. 1. Diagram of the real power source

$$u = e - Zi.$$

Signals u, e and i included in the equation are elements of the same Hilbert space, while Z is the convolution operator which, depending on the type of space, is defined as:

$$Zi(t) = \int_{-\infty}^{\infty} Zi(t - t')i(t')dt'$$

for non–periodic signals, or:

$$Zi(t) = \int_{0}^{T} Z(t \ominus t')i(t')dt'$$

for T-periodic signals. The $\ominus$ character means the modulo T subtraction. In a non-periodic and periodic signal space, the scalar product is defined and in the time domain it takes the following form:

$$(u, i) = \int_{-\infty}^{\infty} u(t)i(t)dt$$

or:

$$(u, i) = \frac{1}{T}\int_{0}^{T} u(t)i(t)dt.$$

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Manuscript submitted 2017-09-25, revised 2017-11-09 and 2018-01-30, initially accepted for publication 2018-02-08, published in October 2018.

The power source balance represented as a scalar product has the following form:

$$(u, i) = (e, i) - (Ri, i),$$

where a positively defined Hermitian operator R has the following form:

$$R = \frac{1}{2}(Z + Z^{*}),$$

where: Z – the source's inner impedance operator, Z^{*} – conjugated operator. In the s-domain, the R operator is defined as:

$$R(s) = \frac{1}{2}[Z(s) + Z(-s)]$$

and in the time domain as:

$$R(t) = \frac{1}{2}[Z(t) + Z(-t)]$$

or:

$$R(t) = \frac{1}{2}[Z(t) + Z(T-t)] \quad \text{for} \quad t \in [0, T)$$

depending on whether it operates in a non-periodic or periodic signal space.

The maximum issue, called "P_{MAX} task", consists in finding the current signal $i(t)$ as to meet the condition below:

$$(e, i) - (Ri, i) \to MAX,$$

which can also be formulated as a minimum task:

$$(Ri, i) - (e, i) \to MIN. \tag{1}$$

The next two minimum tasks are also to find the current signal $i(t)$ so as to:

$$(i, i) \to MIN \tag{2}$$

and:

$$(Ri, i) \to MIN \tag{3}$$

under the common power balance condition:

$$(e, i) - (Ri, i) - P = 0. \tag{4}$$

Tasks (2) and (4), referred to as the "I_{MIN} tasks", involve minimizing the RMS current value under a given active power P delivered from the source, whereas task (3) and (4), known as the "ΔP_{MIN} tasks", minimize the source internal power loss additionally under the condition of a given active power value being delivered from the source.

The conditional task I_{MIN} can be equivalently formulated using Lagrange's scalar factor λ as follows:

$$\begin{aligned}&(i, i) - \lambda[(e, i) - (Ri, i) - P] = \\ &= (i, i) + (\lambda Ri, i) - (\lambda e, i) + \cancel{\lambda P}.\end{aligned}$$

The λP component is irrelevant from the point of view of the current gradient. That is why it is ignored at this stage (it will disappear during calculation of the current gradient anyway). It will re-emerge with the energy functions (chapter 4). Therefore, the I_{MIN} task is formulated as follows:

$$\big((1 + \lambda R)i, i\big) - (\lambda e, i) \to MIN, \tag{5}$$

similarly, the ΔP_{MIN} task is obtained as below:

$$\big((1 + \lambda)Ri, i\big) - (\lambda e, i) \to MIN. \tag{6}$$

Substituting λ with λ^{-1}, yet another form is formulated:

$$\big((\lambda 1 + R)i, i\big) - (e, i) \to MIN \tag{7}$$

for I_{MIN} and:

$$\big((\lambda + 1)Ri, i\big) - (e, i) \to MIN \tag{8}$$

for ΔP_{MIN}.

It is easy to notice that formulas (5) and (6), i.e. I_{MIN} and ΔP_{MIN} become the P_{MAX} task (1) when $\lambda \to \infty$. Similarly, formulas (7) and (8) also become the P_{MAX} task but with $\lambda \to 0$.

3. Solving equations for minimum tasks

Using gradient methods, the solving equations for individual minimum tasks can be obtained. Thus the variation of function (1) with respect to the requested solution $i(t)$ is calculated:

$$\begin{aligned}&\delta[(Ri, i) - (e, i)] = \\ &= (R\delta i, i) + (Ri, \delta i) - (R\delta i, \delta i) - (e, \delta i) = \\ &= (2Ri - e, \delta i) + (R\delta i, \delta i).\end{aligned}$$

Since the functional minimum is sought, for each δi increment it has to meet the following inequality:

$$(2Ri - e, \delta i) + (R\delta i, \delta i) > 0.$$

Since the quadratic form $(R\delta i, \delta i)$ always has a positive sign and $(2Ri - e, \delta i)$ does not preserve the sign, the second form should be reduced to 0 (the zero operator). Thus, from the following condition:

$$(2Ri - e, \delta i) = 0$$

the solution of the P_{MAX} issue is obtained:

$$Ri_d = \frac{1}{2}e. \tag{9}$$

Operating analogously, the solving equations for the I_{MIN} task can be found:

$$(1 + \lambda R)I_{\lambda} = \lambda Ri_d, \tag{10}$$

The same can be done for the ΔP_{MIN} task:

$$(1 + \lambda r) i_\lambda = \lambda r i_d. \tag{11}$$

The current signal $i_d(t)$, which is obtained from operator-equation (9), referred to as “the adjustment current”, appears in operator-equations (10) and (11) as a predetermined function, and from that equations the groups of currents $I_\lambda(t)$ and $i_\lambda(t)$, parameterized with real Lagrange’s factor λ, are obtained. The scalar r in (11) will be called “the source’s normative resistance” [17]. Equations (10) and (11) are the “similar equations”. The linear operator R is replaced with a positive scalar r. This is the “similarity principle” of equations that solves the minimum task I_{MIN} and ΔP_{MIN}.

The solving equation of the P_{MAX} task is the limit equation for the I_{MIN} and ΔP_{MIN} tasks, which is shown in Fig. 2.

$$P_{MAX}: \quad R i_d = \frac{1}{2} e$$

$$I_{MIN}: \quad (1 + \lambda R) I_\lambda = \lambda R i_d$$

$$\lambda \to \infty$$

$$\Delta P_{MIN}: \quad (1 + \lambda r) i_\lambda = \lambda r i_d$$

Fig. 2. Comparison of the solving equations

This means that for $\lambda \to \infty$ solving equations (10) and (11) have solutions equal to the solution of equation (9), i.e. $I_\lambda = i_d$ and $i_\lambda = i_d$.

Note: in the I_{MIN} solving equation the 1 character indicates the identity operator while in the ΔP_{MIN} it is a scalar value.

4. Power functions

Power functions have been called the functions dependent on λ, defined as follows:

$$F(\lambda) = (e, I_\lambda) - (R I_\lambda, I_\lambda) = \big(R(2 i_d - I_\lambda), I_\lambda\big), \tag{12}$$

$$f(\lambda) = (e, i_\lambda) - (R i_\lambda, i_\lambda) = \big(R(2 i_d - i_\lambda), i_\lambda\big), \tag{13}$$

$$\phi(\lambda) = r(2 i_d - I_\lambda, I_\lambda), \tag{14}$$

$$\varphi(\lambda) = r(2 i_d - i_\lambda, i_\lambda). \tag{15}$$

Power functions are used for:

– finding the specific current signal $I_\lambda(t)$ or $i_\lambda(t)$ corresponding to the given power P, which is determined by the following equations:

$$F(\lambda) = P \;\text{ or }\; f(\lambda) = P, \tag{16}$$

– determining the value of the source’s normative resistance r.

In order to investigate power functions, their values and derivatives at zero and at infinity must be calculated. Taking into consideration that functional derivatives of current signals $I'_\lambda = dI_\lambda/d\lambda$, $i'_\lambda = di_\lambda/d\lambda$ meet the following operator-equations:

$$(1 + \lambda R) I'_\lambda = R(i_d - I_\lambda) \tag{17}$$

$$(1 + \lambda r) i'_\lambda = r(i_d - i_\lambda) \tag{18}$$

the λ–based derivatives of power functions will have the forms below:

$$\begin{aligned} F'(\lambda) &= 2\big(R(i_d - I_\lambda), I'_\lambda\big) = \big((1 + \lambda R) I'_\lambda, I'_\lambda\big) \\ f'(\lambda) &= 2\big(R(i_d - i_\lambda), i'_\lambda\big) = 2\,\frac{1 + \lambda r}{r}\,(R i'_\lambda, i'_\lambda) \\ \phi'(\lambda) &= 2r(i_d - I_\lambda, I'_\lambda) = 2r\big(R^{-1}(1 + \lambda R) I'_\lambda, I'_\lambda\big) \\ \varphi'(\lambda) &= 2r(i_d - i_\lambda, i'_\lambda) = 2(1 + \lambda r)\, i'_\lambda, i'_\lambda\big). \end{aligned} \tag{19}$$

All power functions have a common feature: their derivatives are positive-defined square forms for $\lambda \geq 0$, that is due to a positively defined R operator. Therefore they are monotonically increasing with respect to $\lambda \geq 0$.

Furthermore, from (17) and (18) the boundary current signals are obtained: $I_{\lambda=0} = i_{\lambda=0} = 0$ (the zero signal), $I'_{\lambda=0} = R i_d$, $i'_{\lambda=0} = r i_d$, $I_{\lambda\to\infty} = i_{\lambda\to\infty} = i_d$, $I'_{\lambda\to\infty} = i'_{\lambda\to\infty} = 0$ (the zero signal). Therefore boundary values of the power functions are:

$$\begin{aligned} &F(0) = f(0) = \phi(0) = \varphi(0) = 0 \\ &F(\infty) = f(\infty) = (R i_d, i_d) = \frac{1}{4}\big(R^{-1} e, e\big) = P_{MAX} \\ &\phi(\infty) = \varphi(\infty) = r(i_d, i_d), \end{aligned} \tag{20}$$

and the boundary values of their derivatives are:

$$\begin{aligned} &F'(0) = 2(R i_d, R i_d), \\ &f'(0) = \phi'(0) = 2r(R i_d, i_d), \\ &\varphi'(0) = 2r^2(i_d, i_d) \\ &F'(\infty) = f'(\infty) = \phi'(\infty) = \varphi'(\infty) = 0. \end{aligned}$$

With a variety of “concatenation” relations between the power functions, three different variants of normative resistance are obtained:

– for $F'(0) = f'(0)$ [16]:

$$r = \frac{(R i_d, R i_d)}{(R i_d, i_d)} = \frac{(e, e)}{(R^{-1} e, e)}, \tag{21}$$

– for $\phi'(0) = \varphi'(0)$:

$$r = \frac{(R i_d, i_d)}{(i_d, i_d)} = \frac{(R^{-1} e, e)}{(R^{-1} e, R^{-1} e)}, \tag{22}$$

– for $F'(0) = \varphi'(0)$:

$$r = \sqrt{\frac{(Ri_d, Ri_d)}{(i_d, i_d)}} = \sqrt{\frac{(e, e)}{(R^{-1}e, R^{-1}e)}}. \tag{23}$$

The source's normative resistance, according to a similarity principle, allows to replace the source's internal loss operator, which in some cases may even be a large size matrix, with a positively defined scalar. This scalar contains information about the optimization tasks (P_{MAX}, ΔP_{MIN}) used and it constitutes resistance in the physical sense. The obtained values of normative resistances are generally not equal. To prove this, they were calculated for two harmonic distributions (numbered as n and m):

$$\frac{(e, e)}{(R^{-1}e, e)} = R_n \frac{1 + \left(\frac{|E_m|}{|E_n|}\right)^2}{1 + \frac{R_n}{R_m}\left(\frac{|E_m|}{|E_n|}\right)^2},$$

$$\frac{(R^{-1}e, e)}{(R^{-1}e, R^{-1}e)} = R_n \frac{1 + \frac{R_n}{R_m}\left(\frac{|E_m|}{|E_n|}\right)^2}{1 + \left(\frac{R_n}{R_m}\right)^2\left(\frac{|E_m|}{|E_n|}\right)^2},$$

$$\sqrt{\frac{(e, e)}{(R^{-1}e, R^{-1}e)}} = R_n \sqrt{\frac{1 + \left(\frac{|E_m|}{|E_n|}\right)^2}{1 + \left(\frac{R_n}{R_m}\right)^2\left(\frac{|E_m|}{|E_n|}\right)^2}}.$$

In this particular case these values are equal for one-harmonic or fixed-harmonic distributions $\left(R_n = R_m, |E_n| = |E_m|\right)$.

5. Solutions for power $F(\lambda) = P$ and $f(\lambda) = P$ equations

The solutions of equations $F(\lambda) = P$ or $f(\lambda) = P$, the so-called energy equations, will allow to obtain specific currents $I_\lambda(t)$ and $i_\lambda(t)$ as well as the so-called optimal I_{opt} and i_{opt} currents. These signals are not equal but they are similar because they were derived from the equations that solve similar problems [16].

The equation $F(\lambda) = P$ cannot be solved directly. An iterative procedure is required, e.g. the Newton's method, where the solution, if obtained by the iterative function:

$$\Gamma(\lambda) = \lambda + \frac{P - F(\lambda)}{F'(\lambda)}$$

as a sequence:

$$\lambda_{n+1} = \Gamma(\lambda_n). \tag{24}$$

Convergence of the Newton's procedure is determined by a derivative of the iterative function, which is defined as:

$$\Gamma'(\lambda) = \left[F(\lambda) - P\right]\frac{F''(\lambda)}{\left[F'(\lambda)\right]^2}.$$

The second derivative $F''(\lambda)$ has the following form [15]:

$$F''(\lambda) = -6\left(RI'_\lambda, I'_\lambda\right)$$

which is the negative-defined quadratic form. Therefore the $\Gamma(\lambda)$ function has the shape which is shown in Fig. 3 and thus iteration (24) converges to λ_*, i.e. to the solution of the equation $F(\lambda) = P$.

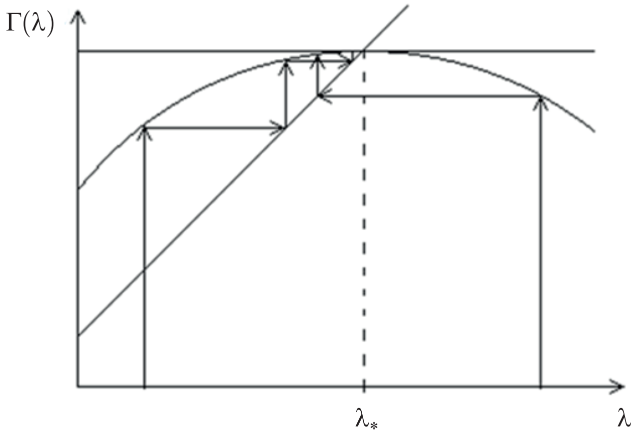

Fig. 3. Graph of iterative function and convergence of Newton's procedure for equation $F(\lambda) = P$

On the other hand, the equation $f(\lambda) = P$ can be solved directly. We use (11):

$$(1 + \lambda r)i_\lambda = \lambda r i_d \Rightarrow i_\lambda = \frac{\lambda r}{1 + \lambda r} i_d$$

in (13):

$$\begin{aligned} f(\lambda) &= \left(R\left(2i_d - \frac{\lambda r}{1 + \lambda r} i_d\right), \frac{\lambda r}{1 + \lambda r} i_d\right) = \\ &= \left(R\left(2 - \frac{\lambda r}{1 + \lambda r}\right) i_d, \frac{\lambda r}{1 + \lambda r} i_d\right) = \\ &= \left(2 - \frac{\lambda r}{1 + \lambda r}\right)\frac{\lambda r}{1 + \lambda r}(Ri_d, i_d) \end{aligned}$$

and according to (20) the following function approximation can be obtained:

$$f(\lambda) = \frac{(2 + \lambda r)\lambda r}{(1 + \lambda r)^2} P_{MAX} \tag{25}$$

which has the following solution:

$$\lambda_*^{-1} = \frac{r}{x}\sqrt{1 - x}\,\left(1 + \sqrt{1 - x}\right) \tag{26}$$

where $x = P/P_{MAX}$ is the so-called fraction of the source's load.

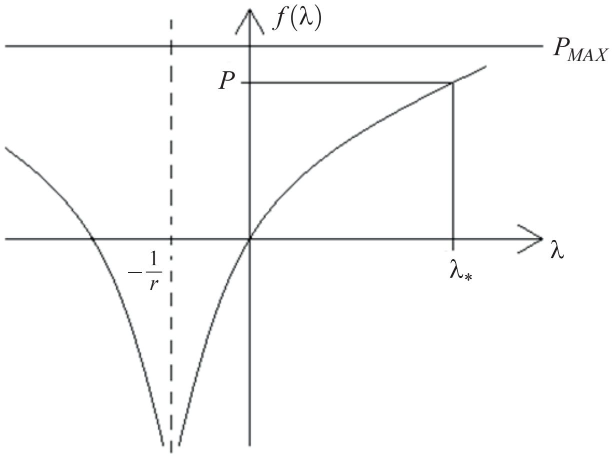

Fig. 4. Power function's approximation

Figure 4 shows the power function's approximation (25).

The transition of the functional (5) to its alternate form (7) takes place by substituting $\lambda \to \lambda^{-1}$ that convert the power function, defined by (13) and (25), according to the transformation formula $f\left(\lambda^{-1}\right) \to f(\lambda)$ and get the following form:

$$f(\lambda) = \frac{(2\lambda + r)r}{(\lambda + r)^2} P_{MAX} \tag{27}$$

which is shown in Fig. 5.

To solve $f(\lambda) = P$, the source's maximum power has to be determined firstly:

$$P_{MAX} = \frac{1}{4}\left(R^{-1}e, e\right), \tag{28}$$

and then the fraction of the source's load has to be calculated:

$$x = \frac{P}{P_{MAX}}.$$

Next the source's normative resistance must be obtained according to (21), (22) or (23), and afterwards the Lagrange's factor is calculated:

$$\lambda_* = r^{-1}\frac{1-\sqrt{1-x}}{\sqrt{1-x}},$$

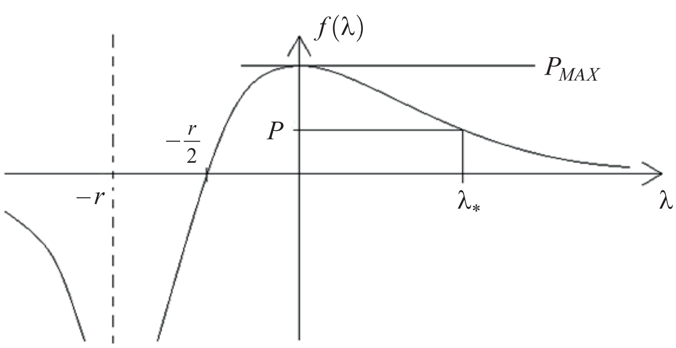

Fig. 5. Alternative power function's approximation

or in the alternate form, with the limitation that normative resistance can be calculated only using equation (22):

$$\lambda_* = r\frac{\sqrt{1-x}}{1-\sqrt{1-x}}.$$

The specific optimal current can now be obtained from the following formula:

$$I_{opt} = \frac{1}{2}\left(\lambda_*^{-1}1 + R\right)^{-1} e. \tag{29}$$

To find the solution to $F(\lambda) = P$, the source's maximum power (28) is to be determined firstly, then the solution λ_* is obtained using iterative method (24), which is finally used to calculate the optimal current (29).

6. Example

In order to compare the methods of searching for optimal solution (10) and (11), an example was used. For the periodic energy source shown in Fig. 6, the minimal RMS source's current $i_{MIN}(t)$, which carries given power $P = 700\,[W]$, should be found.

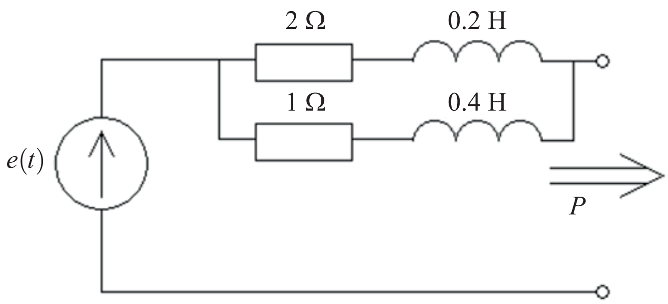

Fig. 6. Loss at the energy source

The source's electromotive force is given (Fig. 7):

$$e(t) = 50\sqrt{2}\cos(2\pi t) - 30\sqrt{2}\sin(6\pi t) + 10\sqrt{2}\cos(10\pi t)\,[V].$$

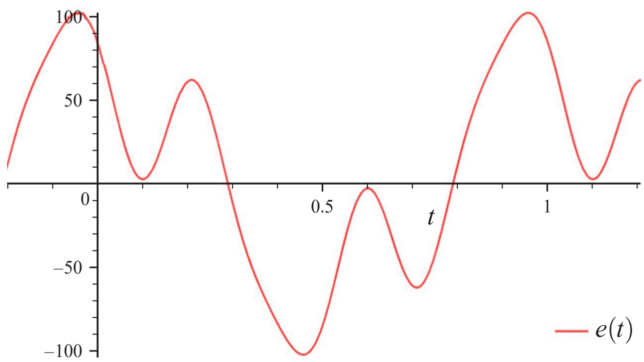

Fig. 7. The source's voltage $e(t)$

The source's internal impedance operator is also determined:

$$Z(s) = \frac{(2+0.2\,s)(1+0.4\,s)}{3+0.6\,s}.$$

On its basis, the Hermitian operator of the source's internal loss is obtained:

$$R(s) = \frac{1}{2}\left[Z(s)+Z(-s)\right] = \frac{-2+0.12\,s^2}{-3+0.12\,s^2}.$$

The maximum power that the source could deliver is:

$$P_{MAX} = \frac{1}{4}(R^{-1}e, e) = \frac{1}{4T}\int_0^T R^{-1}e(t)e(t)dt = 997.72,$$

where:

$$R^{-1}e(t) = \int_0^T R^{-1}(t\ominus\tau)e(\tau)d\tau = \int_0^t R^{-1}(t-\tau)e(\tau)d\tau + {} + \int_t^T R^{-1}(t-\tau+T)e(\tau)d\tau.$$

The adjustment current, i.e. delivering the maximum power, is obtained from (9) and its waveform is shown in Fig. 8.

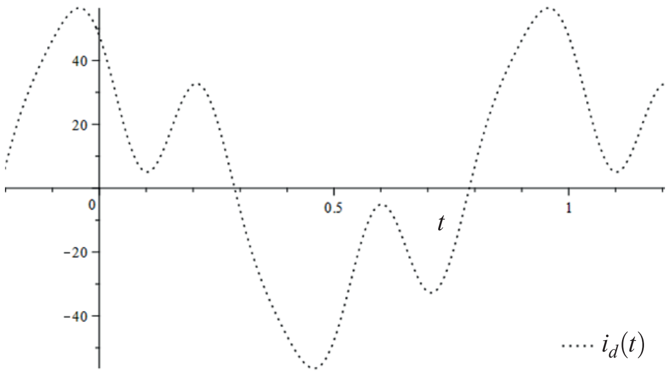

Fig. 8. Adjustment current $i_d(t)$

The energy source must deliver power $P = 700\,[W]$ and on this basis the fraction of the source's load is calculated:

$$x = \frac{P}{P_{MAX}} = 0.7016027403.$$

Based on the "principle of similarity" [17], the I_{MIN} task is replaced with the ΔP_{MIN} task. It allows to replace the source's inner loss operator R with "the source's normative resistance" of a scalar value. Therefore, according to (21–23):

$$r_1 = \frac{(e,e)}{(R^{-1}e,e)} = \frac{(e,e)}{4P_{MAX}} = 0.8770034256,$$

$$r_2 = \frac{(R^{-1}e,e)}{(R^{-1}e,R^{-1}e)} = 0.6551817364,$$

$$r_3 = \sqrt{\frac{(e,e)}{(R^{-1}e,R^{-1}e)}} = 0.7580215216.$$

Then the energy function approximation is used:

$$f(\lambda) = \frac{(2+\lambda r)\lambda r}{(1+\lambda r)^2} P_{MAX} = P$$

to obtain the Lagrange's λ factor corresponding to the required power delivery by the energy source, hence:

$$\lambda_1 = (r_1)^{-1}\frac{1-\sqrt{1-x}}{\sqrt{1-x}} = 0.9471325266,$$

$$\lambda_2 = (r_2)^{-1}\frac{1-\sqrt{1-x}}{\sqrt{1-x}} = 1.267798573,$$

$$\lambda_3 = (r_3)^{-1}\frac{1-\sqrt{1-x}}{\sqrt{1-x}} = 1.095798005.$$

Using the new energy function approximation:

$$f(\lambda) = \frac{(2\lambda+r)r}{(\lambda+r)^2} P_{MAX} = P,$$

with the proviso that the source's normative resistance must be calculated from formula (22), the next Lagrange's factor is obtained:

$$\lambda_4 = r_2\frac{\sqrt{1-x}}{1-\sqrt{1-x}} = 0.7887688322.$$

Finally, the optimal current (which minimizes the source's inner losses) is obtained through transformed formula (11) with replacement of the source's internal loss operator R by normative resistance r:

$$i_{MIN}(t) = i^1_{opt}(t) = \frac{\lambda_1 r_1}{1+\lambda_1 r_1} i_d(t).$$

Its waveform is shown in Fig. 9.

Figure 10 summarizes waveforms of the source's voltage $e(t)$, the adjustment current (which provides maximum power from the source) $i_d(t)$ and optimal current $i^1_{opt}(t)$, which delivers the required power to the source's output terminals.

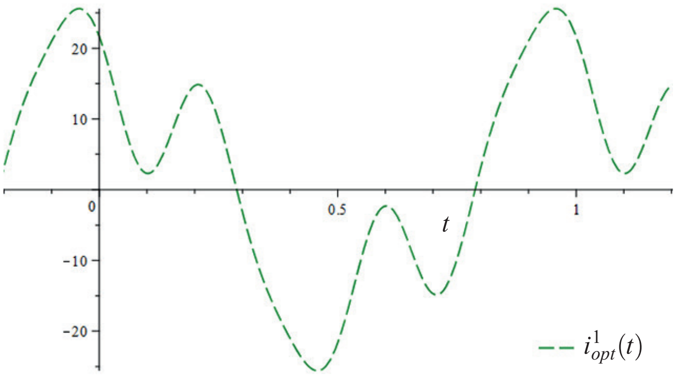

Fig. 9. Optimal current $i^1_{opt}(t)$

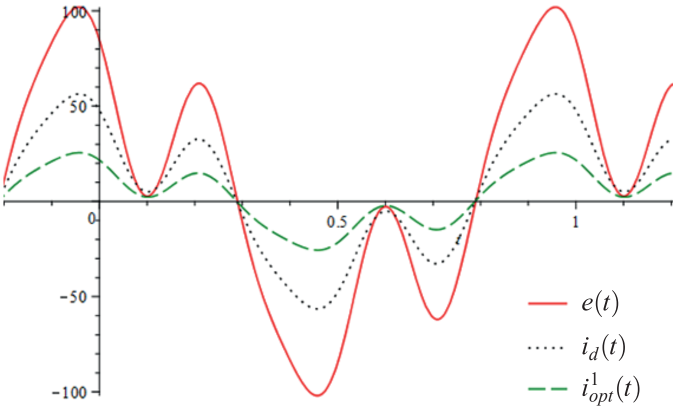

Fig. 10. Summary of the source's voltage, adjustment current and optimal current

The other optimal current waveforms (for other λ and r values) are almost identical, therefore instead of presenting them, the absolute error waveforms, referred to the first optimal current, were shown (Fig. 11–13).

In order to verify effectiveness of the approximation methods, the optimal current was calculated with exact method (I_{MIN} task) and compared with the currents presented above. The approximation methods allow for instant (in the operator sense) determination of optimal current for the ΔP_{MIN} criterion, while the exact method is used to calculate optimal current for the I_{MIN} criterion. A comparison of these methods (providing individual but similar solutions [17]) is intended to show with what accuracy it is possible to replace a complex (in the operator sense) I_{MIN} task with a much faster, similar ΔP_{MIN} task.

In order to solve the I_{MIN} task, power function (12) parameterized with Lagrange's factor λ was calculated:

$$\begin{aligned} F(\lambda) &= (e, I_\lambda) - (RI_\lambda, I_\lambda) = \\ &= \frac{1}{2}\lambda\left(e, (1+\lambda R)^{-1}e\right) - \\ &- \frac{1}{4}\lambda^2\left(R(1+\lambda R)^{-1}e, (1+\lambda R)^{-1}e\right) = P, \end{aligned} \tag{30}$$

where:

$$\begin{aligned} (1+\lambda R)^{-1}e(t) &= \int_0^t (1+\lambda R)^{-1}(t-\tau)e(\tau)d\tau + \\ &+ \int_t^T (1+\lambda R)^{-1}(t-\tau+T)e(\tau)d\tau, \end{aligned} \tag{31}$$

$$\left(e, (1+\lambda R)^{-1}e\right) = \frac{1}{T}\int_0^T e(t)\left[(1+\lambda R)^{-1}e\right](t)dt. \tag{32}$$

The function $(1+\lambda R)^{-1}(t)$ in (31) was obtained using operational calculus [5, 19], i.e. based on the inner loss operator $R(s)$, the $(1+\lambda R)(s)$ operator was created, then inverted and finally the $(1+\lambda R)^{-1}(s)$ operator was converted in the time domain.

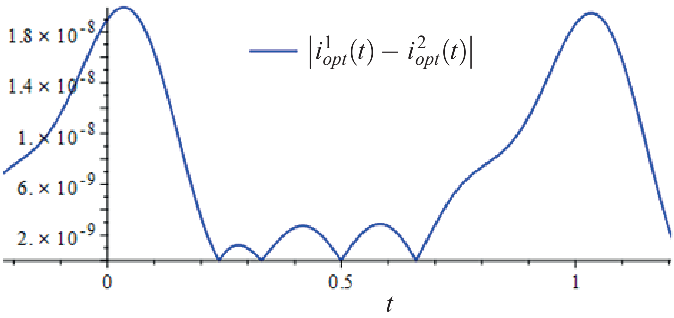

Fig. 11. Absolute error waveform $\left|i^1_{opt}(t) - i^2_{opt}(t)\right|$

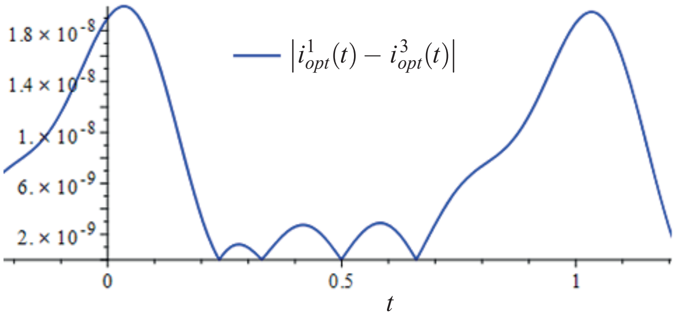

Fig. 12. Absolute error waveform $\left|i^1_{opt}(t) - i^3_{opt}(t)\right|$

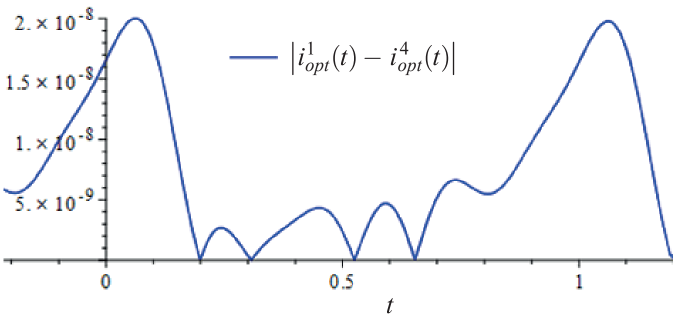

Fig. 13. Absolute error waveform $\left|i^1_{opt}(t) - i^4_{opt}(t)\right|$

Other operators, convolution integrals and scalar product integrals in (30) were calculated similarly as in (31) and (32). The given power $P = 700\,[W]$ was assumed and the power function was solved using the Newton's procedure in order to acquire Lagrange's factor:

$$\lambda = 1.254850931.$$

Finally, the optimal current was obtained:

$$\begin{aligned} I_{opt}(t) &= \frac{1}{2}\lambda(1+\lambda R)^{-1}e(t) = \\ &= \frac{1}{2}\lambda\left[\int_0^t (1+\lambda R)^{-1}(t-\tau)e(\tau)d\tau + \right. \\ &\left. + \int_t^T (1+\lambda R)^{-1}(t-\tau+T)e(\tau)d\tau\right], \end{aligned}$$

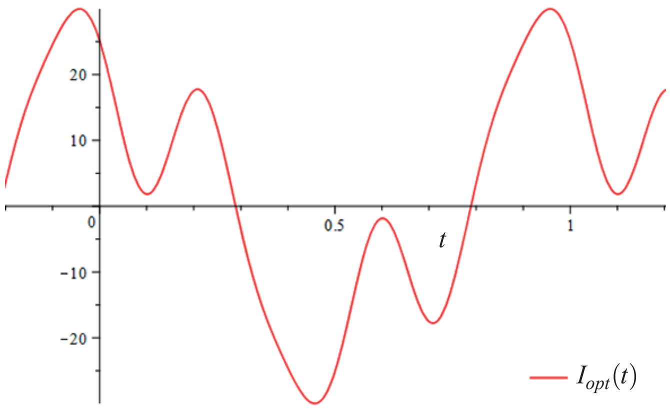

Fig. 14. Optimal current obtained using the exact method

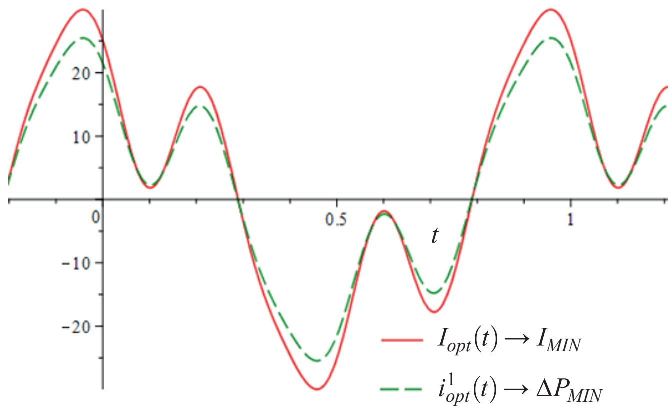

Fig. 15. Comparison of the optimal currents

Its waveform is shown in Fig. 14.

A comparison of the optimal current obtained using the approximate method $i^1_{opt}(t)$ (one of them because of their almost identical waveforms) and the one obtained using Newton's procedure $I_{opt}(t)$ is shown in Fig. 15.

In the example, the minimum current supplying the given power P to the source's terminals should have been determined. According to the principle of similarity [17], the current causing minimal source's inner losses and supplying the given power P to the source's terminals was calculated. It has been achieved through the application of a new and previously known approximation in the optimization algorithms. The first optimal current was obtained with the use of an algorithm from the author's publication [17], while the remaining currents were calculated according to the new algorithms. The presented results, with absolute errors within the limits of 10^{-8}, confirm the effectiveness of the new solutions as compared to the previously known ones. The result obtained using the Newton's procedure is accurate (according to the assumptions of the example) but much more complicated than the approximation methods. The divergence with the average below 15% is acceptable especially when it comes to calculation time. Even in a simple example, the optimal current was obtained almost instantly when using the approximate methods while the exact method took over a dozen hours. In practice, such calculation time is unacceptable for obtaining an optimal solution. The advantage of the approximation method is the replacement of the source's inner loss operator with a number which significantly reduces computation time. Therefore the approximation methods are much more efficient, despite their less accurate results. However, both solutions are optimal from the perspective of two similar optimization criteria. It should also be noted that the 15% difference is not a universal quantity (it is related to this concrete example), and the value will be different for each two-terminal circuit.

7. Conclusion

In the paper some new results, as compared to those previously known, were obtained. New approximations of the so-called power functions have been introduced in the form of relations between the given source's power and the Lagrange's factor. Assuming various "concatenating" criteria of the power functions, the source's normative resistance has been obtained and the differences between them has been highlighted. A comparison of the individual minimum tasks for the power sources have been made. In order to illustrate the approximation methods presented in the article, an example was calculated and the methods were compared.

This paper is designed to extend the state of existing knowledge in the source-receiver matching field. Previous studies do not use optimization techniques, even those that are implemented using gradient methods. In existing studies, models of a voltage-stiff source, i.e. one not taking account of the interrelations between voltage and the current signals, or of a lossless source at the most, are being used. Effective conductance is also used, as determined by Fryze. It is associated with the minimum current supplying the given power to the source's terminals, where the source described by Fryze is a voltage-stiff source [1, 2, 8, 12, 23, 25, 27–32]. The inner impedance of such source is a zero operator. This trivializes the task of matching the receiver with the source. The energy function built on this basis is a linear function.

The presented study uses optimization methods in a systematic way and introduces a non-zero loss operator. This greatly improves the source-receiver matching theory in such sense that a non-trivial energy function $F(\lambda)$ appears. Application of the power source with inner loss operator R in the power theory allows to formulate additional optimization criteria that have not been used to date. It should also be noted that effective conductance is a special case for a voltage-stiff source, therefore this article is an extension and generalization for a real voltage source. The principle of similarity introduced in the article acts as a model for a source that is located between a trivial voltage-stiff source and a source with a non-zero loss operator.

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