10.24425/acs.2018.125481

Archives of Control Sciences
Volume 28(LXIV), 2018
No. 4, *pages 511–525*

A new method for computation of positive realizations of fractional linear continuous-time systems

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A new method for computation of positive realizations of given transfer matrices of fractional linear continuous-time linear systems is proposed. Necessary and sufficient conditions for the existence of positive realizations of transfer matrices are given. A procedure for computation of the positive realizations is proposed and illustrated by examples.

Key words: determination, positive, realization, transfer matrix, linear, continuous-time, fractional, system

1. Introduction

A dynamical system is called fractional if it is described by fractional order differential equation [28, 31–34]. The fundamentals of fractional differential equations and systems have been given in [31–34]. The stability of fractional linear systems have been analyzed in [3–5].

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems theory is given in the monographs [2, 6, 16].

Positive linear systems with different fractional orders have been addressed in [17, 18, 35]. Descriptor (singular) fractional linear systems have been analyzed in [19]. The stability of a class of nonlinear fractional-order systems has been analyzed [10, 27, 36].

The determination of the matrices A, B, C, D of the state equations of linear systems for given their transfer matrices is called the realization problem.

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The studies have been carried out in the framework of work No. S/WE/1/2016 and financed from the funds for science by the Polish Ministry of Science and Higher Education.

Rceived 30.12.2017.

The realization problem is a classical problem of analysis of linear systems and has been considered in many books and papers [8, 9, 15, 28, 30]. A tutorial on the positive realization problem has been given in the paper [1] and in the books [6, 16]. The positive minimal realization problem for linear systems without and with delays has been analyzed in [7, 11–13, 16, 20–23, 26, 27, 29]. The existence and determination of the set of Metzler matrices for given stable polynomials have been considered in [14]. The realization problem for positive 2D hybrid systems has been addressed in [25]. For fractional linear systems the realization problem has been considered in [24, 28, 30].

In this paper a new method for computation of positive realizations of linear continuous-time systems is proposed.

The paper is organized as follows. In section 2 some definitions and theorems concerning the positive and fractional continuous-time linear systems are recalled. A new method for computation of positive realizations for single-input single-output linear systems is proposed in section 3 and for multi-input multi-output systems in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Preliminaries

In this paper the following Caputo definition of the fractional derivative of α order will be used [28]

$$
{}_0D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int\limits_0^t \frac{\dot{f}(\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau, \quad 0<\alpha<1, \tag{1}
$$

where $\dot{f}(\tau) = \dfrac{\mathrm{d}f(\tau)}{\mathrm{d}\tau}$ and $\Gamma(x) = \displaystyle\int\limits_0^\infty t^{x-1}e^{-t}\mathrm{d}t$, $\mathrm{Re}(x) > 0$ is the Euler gamma function.

Consider the fractional continuous-time linear system

$$
\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0<\alpha<1, \tag{2a}
$$

$$
y(t) = Cx(t) + Du(t), \tag{2b}
$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Definition 1 [28] *The fractional system (2) is called (internally) positive if* $x(t) \in \Re_+^n$ *and* $y(t) \in \Re_+^p$, $t \geqslant 0$ *for any initial conditions* $x(0) \in \Re_+^n$ *and all inputs* $u(t) \in \Re_+^m$, $t \geqslant 0$.

Theorem 1 [28] *The fractional system (2) is positive if and only if*

$$A \in M_n, \qquad B \in \Re_+^{n\times m}, \qquad C \in \Re_+^{p\times n}, \qquad D \in \Re_+^{p\times m}. \tag{3}$$

Definition 2 [28] *The fractional positive system (2) for* $u(t) = 0$ *is called asymptotically stable if*

$$\lim_{t\to\infty} x(t) = 0 \quad \textit{for all} \;\; x(0) \in \Re_+^n. \tag{4}$$

Theorem 2 [28] *The fractional positive system (2) for* $u(t) = 0$ *is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

1. *The eigenvalues* λ_i, $i = 1,\ldots,n$ *of the matrix* $A \in M_n$ *satisfy the condition*

$$\mathrm{Re}\lambda_i < 0, \quad i = 1,\ldots,n. \tag{5}$$

2. *All coefficients of the characteristic polynomial*

$$\det[I_n w - A] = w^n + a_{n-1}w^{n-1} + \ldots + a_1 w + a_0, \quad w = s^\alpha \tag{6}$$

are positive, i.e. $a_k > 0$ *for* $k = 0,1,\ldots,n-1$.

3. *All principal minors* $\overline{M}_i$, $i = 1,\ldots,n$ *of the matrix* $-A$ *are positive, i.e.*

$$\overline{M}_1 = |-a_{11}| > 0, \quad \overline{M}_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \;\ldots,\; \overline{M}_n = \det[-A] > 0. \tag{7}$$

4. *There exists strictly positive vector* $\lambda^T = [\lambda_1 \;\cdots\; \lambda_n]^T$, $\lambda_k > 0$, $k = 1,\ldots,n$ *such that*

$$A\lambda < 0 \quad \textit{or} \quad A^T\lambda < 0 \tag{8}$$

If $\det A \neq 0$ then we may choose $\lambda = A^{-1}c$, where $c \in \Re^n$ is any strictly positive vector.

The transfer matrix of the system (2) is given by

$$T(s) = C[I_n w - A]^{-1}B + D, \qquad w = s^\alpha. \tag{9}$$

The transfer matrix is called proper if

$$\lim_{w\to\infty} T(w) = D \in \Re_+^{p\times m} \tag{10}$$

and it is called strictly proper if $D = 0$.

Definition 3 [6, 30] *The matrices (3) are called a positive realization of* $T(w)$ *if they satisfy the equality (9).*

Definition 4 [6, 30] *The realization (3) is called asymptotically stable if the matrix* A *is an asymptotically stable Metzler matrix (Hurwitz Metzler matrix).*

Theorem 3 [30] *If (3) is a positive realization of (9) then the matrices*

$$\overline{A} = PAP^{-1}, \qquad \overline{B} = PB, \qquad \overline{C} = CP^{-1}, \qquad \overline{D} = D \tag{11}$$

are also a positive realization of (9) if and only if the matrix $P \in \Re_+^{n\times n}$ *is a monomial matrix (in each row and in each column only one entry is positive and the remaining entries are zero).*

Proof. Proof follows immediately from the fact that $P^{-1} \in \Re_+^{n\times n}$ if and only if P is a monomial matrix. □

3. Positive realizations of transfer functions

In this section necessary and sufficient conditions will be given for the existence of positive realizations (A, B, C, D) of the given transfer function

$$T(w) = \frac{m_n w^n + m_{n-1}w^{n-1} + \ldots + m_1 w + m_0}{w^n + d_{n-1}w^{n-1} + \ldots + d_1 w + d_0}. \tag{12}$$

Using (10) we obtain

$$D = \lim_{w\to\infty} T(w) = m_n \tag{13}$$

and

$$\overline{T}(w) = T(w) - D = \frac{\overline{m}_{n-1}w^{n-1} + \ldots + \overline{m}_1 w + \overline{m}_0}{s^n + d_{n-1}w^{n-1} + \ldots + d_1 w + d_0} = C[I_n w - A]^{-1}B, \tag{14a}$$

where

$$\overline{m}_k = m_k - m_n d_k \qquad \text{for } k = 0, 1, \ldots, n-1. \tag{14b}$$

Theorem 4 *There exists the positive realization*

$$A = \begin{bmatrix} -w_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -w_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -w_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 1 & -w_n \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \tag{15}$$

$$C = [0 \ \cdots \ 0 \ 1], \qquad D = m_n$$

of the transfer function (12) if and only if the following conditions are satisfied:

$$1)\qquad m_n \geqslant 0, \tag{16a}$$

$$2)\qquad S^{-1}M \in \Re_+^n, \tag{16b}$$

where w_k*,* $k = 1,\dots,n$ *are the zeros of the denominator*

$$\begin{aligned} d(w) &= w^n + d_{n-1}w^{n-1} + \dots + d_1 w + d_0 \\ &= (w+w_1)(w+w_2)\dots(w+w_n), \end{aligned} \tag{16c}$$

$$S = \begin{bmatrix} 1 & w_1 & w_1w_2 & \cdots & w_1w_2\dots w_{n-1} \\ 0 & 1 & w_1+w_2 & \cdots & w_1+w_2+\dots+w_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad M = \begin{bmatrix} \overline{m}_0 \\ \overline{m}_1 \\ \vdots \\ \overline{m}_{n-1} \end{bmatrix}, \qquad \overline{m}_0 > 0. \tag{16d}$$

Proof. It is easy to check that

$$\begin{aligned} C[I_n w - A]^{-1} &= C\begin{bmatrix} w+w_1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & w+w_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & w+w_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & -1 & w+w_n \end{bmatrix} \\ &= \frac{C[I_n w - A]_{ad}}{d(w)}, \end{aligned} \tag{17a}$$

where product of the matrix C and the adjoint matrix $[I_n w - A]_{ad}$ has the form

$$\begin{aligned} C[I_n w - A]_{ad} = [1 \;\; w+w_1 \;\; (w+w_1)(w+w_2) \;\; \cdots \\ \dots \;\; (w+w_1)(w+w_2) \;\; \dots \;\; (w+w_{n-1})]. \end{aligned} \tag{17b}$$

Using (17) and (15) we obtain

$$\begin{aligned}
C[I_n w - A]^{-1}B &= \frac{C[I_n w - A]_{ad}B}{d(w)} \\
&= \frac{1}{d(w)}[1 \;\; w+s_1 \;\; (w+w_1)(w+w_2) \;\; \cdots \;\; (w+w_1)(w+w_2)\dots(w+w_{n-1})]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
&= \frac{1}{d(w)}[b_1 + b_2(w+w_1) + b_3(w+w_1)(w+w_2) + \dots \\
&\qquad \dots + b_n(w+w_1)(w+w_2)(w+w_{n-1})] \\
&= \frac{b_1 + b_2 w_1 + b_3 w_1 w_2 + \dots + b_n w_1 w_2 \dots w_{n-1}}{d(w)} \\
&\qquad + \frac{[b_2 + b_3(w_1+w_2) + \dots + b_n(w_1+w_2+\dots+w_{n-1})]w + \dots + b_n w^{n-1}}{d(w)} \\
&= \frac{\overline{m}_{n-1}s^{n-1} + \dots + \overline{m}_1 s + \overline{m}_0}{d(w)} = \overline{T}(w).
\end{aligned} \tag{18}$$

and the matrices S, B and M are related by the equation

$$SB = M. \tag{19}$$

The matrix $B \in \Re_+^n$ if and only if the condition (16b) is satisfied.

Note that the realization (15) is positive if and only if the conditions (16a) and (16b) are satisfied. □

Remark 1 *The positive realization (15) of (12) is asymptotically stable if and only if $d_k > 0$ for $k = 0, 1, \dots, n-1$.*

Proof. By Theorem 2 the zeros w_k of the polynomial $d(w)$ satisfy the condition $Re w_k < 0$ for $k = 1, \dots, n$ if and only if $d_k > 0$ for $k = 0, 1, \dots, n-1$. □

Remark 2 *For the transfer function*

$$T(w) = \frac{m_0}{w^n + d_{n-1}w^{n-1} + \dots + d_1 w + d_0}, \quad m_0 > 0 \tag{20}$$

there always exists the positive realization

$$A = \begin{bmatrix} -w_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -w_2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -w_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -w_n \end{bmatrix}, \qquad B = \begin{bmatrix} m_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{21}$$
$$C = [0 \ \cdots \ 0 \ 1], \qquad D = 0.$$

The positive realization is asymptotically stable if and only if $d_k > 0$ *for* $k = 0, 1, \ldots, n-1$.

Theorem 5 *There exists the positive realization*

$$\overline{A} = \begin{bmatrix} -w_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -w_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -w_{n-1} & 1 \\ 0 & 0 & 0 & \cdots & 0 & -w_n \end{bmatrix}, \qquad \overline{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{22}$$
$$\overline{C} = [c_1 \ c_2 \ \cdots \ c_n], \qquad D = m_n$$

of the transfer function (12) if and only if the conditions (16a) and (16b) are satisfied, where s_k, $k = 1, \ldots, n$ *are the zeros of (16c), S and M are defined by (16d).*

Proof. The proof is similar (dual) to the proof of Theorem 4. □

Example 1 Find the positive realization (15) of the transfer function

$$T(s) = \frac{m_2 w^2 + m_1 w + m_0}{w^2 + d_1 w + d_0} = \frac{3w^2 + 7w - 2}{w^2 + 2w - 3}. \tag{23}$$

Using (13), (14a) and (23) we obtain

$$D = \lim_{w \to \infty} T(w) = \lim_{w \to \infty} \frac{3w^2 + 7w - 2}{w^2 + 2w - 3} = m_2 = 3 \tag{24}$$

and

$$\overline{T}(w) = T(w) - D = \frac{\overline{m}_1 w + \overline{m}_0}{w^2 + d_1 w + d_0} = \frac{w + 7}{w^2 + 2w - 3}, \tag{25}$$

where $\overline{m}_1 = m_1 - m_2 d_1 = 1$, $\overline{m}_0 = m_0 - m_2 d_0 = 7$.

The polynomial

$$d(w) = w^2 + 2w - 3 \tag{26}$$

has the zeros: $w_1 = 1$, $w_2 = -3$ and the matrix A has the form

$$A = \begin{bmatrix} w_1 & 0 \\ 1 & w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -3 \end{bmatrix}. \tag{27}$$

Using (16d) and (25) we obtain

$$B = \begin{bmatrix} 1 & w_1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \overline{m}_0 \\ \overline{m}_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \tag{28}$$

and

$$C = [0 \ \ 0 \ \ 1]. \tag{29}$$

The desired positive realization of (23) is given by (27), (28), (29) and (24).

It is easy to check that the matrices

$$A = \begin{bmatrix} -3 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = [6 \ \ 1], \qquad D = 3 \tag{30}$$

are also the positive realization of the transfer function (23).

4. Positive realizations for multi-input multi-output systems

In this section the method presented in section 3 will be extended to multi-input multi-output (MIMO) linear systems. Without loss of generality and to simplify the notation two-input two-output systems will be considered.

The problem under the considerations can be stated as follows. Given the proper transfer matrix

$$\begin{gathered} T(w) = \begin{bmatrix} T_{11}(w) & T_{12}(w) \\ T_{21}(w) & T_{22}(w) \end{bmatrix}, \\ T_{ik}(w) = \frac{m_{ikn}w^n + \ldots + m_{ik1}w + m_{ik0}}{w^n + d_{ikn-1}w^{n-1} + \ldots + d_{ik1}w + d_{ik0}}, \qquad i,k = 1,2 \end{gathered} \tag{31}$$

find the positive realization (A, B, C, D) such that

$$T(s) = C[I_n w - A]^{-1}B + D. \tag{32}$$

Using

$$D = \lim_{w \to \infty} T(w) \tag{33}$$

we may find the matrix D and the strictly proper transfer matrix

$$\overline{T}(w) = T(w) - D = C[I_n w - A]^{-1}B = \begin{bmatrix} \frac{\overline{m}_{11}(w)}{d_1(w)} & \frac{\overline{m}_{12}(w)}{d_1(w)} \\ \frac{\overline{m}_{21}(w)}{d_2(w)} & \frac{\overline{m}_{22}(w)}{d_2(w)} \end{bmatrix}, \tag{34}$$

where

$$d_i(w) = w^n + d_{in-1}w^{n-1} + \ldots + d_{i1}w + d_{i0}, \qquad i = 1,2 \tag{35a}$$

is the least common denominator of $T_{i1}(w)$ for $i = 1,2$ and w_{i1}, w_{i2}, $\ldots$, w_{in}, $i = 1,2$ are its zeros, i.e.

$$d_i(w) = (w + w_{i1})(w + w_{i2})\ldots(w + w_{in}), \qquad i = 1,2 \tag{35b}$$

and

$$\overline{m}_{ik}(w) = \overline{m}_{ikn-1}w^{n-1} + \ldots + \overline{m}_{ik1}w + \overline{m}_{ik1} + \overline{m}_{ik0}, \qquad i = 1,2. \tag{36}$$

The matrices A_i of the positive realizations have the form

$$A_i = \begin{bmatrix} -w_{i1} & 0 & 0 & \cdots & 0 & 0 \\ 1 & -w_{i2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -w_{in-1} & 0 \\ 0 & 0 & 0 & \cdots & 1 & -w_{in} \end{bmatrix}, \qquad i = 1,2 \tag{37}$$

and

$$A = \text{blockdiag}[A_1 \;\; A_2] = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}. \tag{38}$$

The matrices B and C have the forms

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \qquad B_{ik} = \begin{bmatrix} b_{ik1} \\ b_{ik2} \\ \vdots \\ b_{ikn_i-1} \end{bmatrix} \in \Re_+^{n_i}, \qquad i,k = 1,2 \tag{39}$$

and

$$C = \text{blockdiag}[C_1 \;\; C_2], \qquad C_i = [0 \;\; \cdots \;\; 0 \;\; 1] \in \Re_+^{1\times n_i}, \quad i = 1,2. \tag{40}$$

The entries of B_{ik}, $i,k = 1,2$ are calculated in the same way as the entries of B in section 3 using the equation (19).

Therefore, we have the following theorem.

Theorem 6 *There exists the positive realization given by (33), (38), (39) and (40) of the transfer matrix (31) if and only if the following conditions are satisfied:*

$$1) \qquad D \in \Re_+^{2\times 2} (\textit{defined by (33)}), \tag{41}$$

$$2) \qquad S_i^{-1} M_i \in \Re_+^{n_i}, \quad i = 1,2, \tag{42a}$$

where

$$S_i = \begin{bmatrix} 1 & w_{i1} & w_{i1}w_{i2} & \cdots & w_{i1}w_{i2}\dots w_{in-1} \\ 0 & 1 & w_{i1}+w_{i2} & \cdots & w_{i1}+w_{i2}+\ldots+w_{in-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad i = 1,2, \tag{42b}$$

$$M_{ik} = \begin{bmatrix} \overline{m}_{ik0} \\ \overline{m}_{ik1} \\ \vdots \\ \overline{m}_{ikn_i-1} \end{bmatrix}, \qquad i,k = 1,2. \tag{42c}$$

Proof. The realization is positive if and only if the condition (41) is satisfied. The matrix A defined by (38) and (37) is a Metzler matrix and it is Hurwitz (asymptotically stable) if $Rew_{ik} < 0$ for $i = 1,2$ and $k = 1,\ldots,n$. The matrix $B \in \Re_+^{(n_1+n_2)\times 2}$ if and only if the conditions (42) are satisfied. The matrix C defined by (40) is always nonnegative. Therefore, the realization given by (33), (38), (39) and (40) is positive if and only if the conditions (41) and (42) are satisfied. □

From the above considerations we have the following procedure for computation of the positive realization (A, B, C, D) for given transfer matrix (31).

Procedure 1

Step 1. Knowing $T(w)$ and using (33) and (34) compute the matrix D and the strictly proper transfer matrix $\overline{T}(w)$.

Step 2. Compute the zeros w_{ij}, $i = 1,2$, $j = 1,\ldots,n$ of the polynomial (36) and the matrices (37) and (38).

Step 3. Using (42b) and (42c) compute the matrices S_i and M_{ik}, $i,k = 1,2$ and check the conditions (42a). If the conditions (42a) are satisfied then there exists $B \in \Re_+^{(n_1+n_2)\times 2}$ and the positive realization of the matrix (31).

Step 4. The desired positive realization is given by (38), (39), (40) and (41).

Example 2 Find the positive realization of the transfer matrix

$$T(w) = \begin{bmatrix} \dfrac{w^2+5w+5}{w^2+3w+2} \\ \dfrac{2w+7}{w+3} \end{bmatrix}. \tag{43}$$

Using Procedure 1 we obtain the following:

Step 1. Using (33), (34) and (43) we obtain

$$D = \lim_{w\to\infty} T(w) = \lim_{w\to\infty} \begin{bmatrix} \dfrac{w^2+5w+5}{w^2+3w+2} \\ \dfrac{2w+7}{w+3} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{44}$$

and

$$\overline{T}(w) = T(w) - D = \begin{bmatrix} \dfrac{2w+3}{w^2+3w+2} \\ \dfrac{1}{w+3} \end{bmatrix}. \tag{45}$$

Step 2. The zeros of the polynomial

$$d_1(w) = w^2+3w+2 \tag{46}$$

are $w_{11} = -1$, $w_{12} = -2$ and the polynomial

$$d_2(w) = w+3 \tag{47}$$

has only one zero $w_{21} = -3$.

In this case the matrix (38) has the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}. \tag{48}$$

Step 3. Taking into account that in this case

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix}, \qquad B_2 = [b_{13}] \tag{49}$$

and using the equation (19) we obtain

$$B_1 = \begin{bmatrix} 1 & -w_{11} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \overline{m}_{10} \\ \overline{m}_{11} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{50a}$$

and

$$B_2 = [b_{13}] = 1. \tag{50b}$$

Therefore, the matrix B has the form

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \tag{51}$$

and the matrix

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{52}$$

Step 4. The desired positive realization of (43) is given by (48), (51), (52) and (44).

It is easy to check that the matrices

$$\overline{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{53}$$

are also the (dual) positive realization of the transfer matrix (43).

Remark 3 *To the presented method the dual method based on the common denominator for each column of $T(w)$ can be applied.*

Remark 4 *By Theorem 3 if the matrices A, B, C, D are a positive realization of $T(s)$ then the matrices PAP^{-1}, PB, CP^{-1}, D are also its positive realization for any monomial matrix P.*

5. Concluding remarks

A new method for computation of positive realizations of transfer matrices of fractional linear continuous-time systems has been proposed. Necessary and sufficient conditions for the existence of the positive realizations have been established (Theorems 4, 5 and 6). A procedure for computation of the positive realizations has been proposed and illustrated by an example (Example 4). The presented method can be extended to fractional linear discrete-time systems.

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