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$M/\vec{G}/n/(0,V)$ Erlang queueing system with non-homogeneous customers, non-identical servers and limited memory space

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Abstract. In the present paper, we investigate a multi-server Erlang queueing system with heterogeneous servers, non-homogeneous customers and limited memory space. The arriving customers appear according to a stationary Poisson process and are additionally characterized by some random volume. The service time of the customer depends on his volume and the joint distribution function of the customer volume and his service time can be different for different servers. The total customers volume is limited by some constant value. For the analyzed model, steady-state distribution of number of customers present in the system and loss probability are calculated. An analysis of some special cases and some numerical examples are attached as well.

Key words: multi-server queueing systems, queueing systems with non-homogeneous customers, queueing systems with heterogeneous servers, loss probability, Stieltjes convolution.

1. Introduction

In the classical queueing theory, we usually investigate systems with some finite or infinite number of identical servers having some finite or infinite queues. For such systems, service time distribution is the same for all servers. The main purpose of the analysis is to obtain very important, from the practical point of view, characteristics such as: distribution of number of customers present in the system, waiting time distribution (for systems with finite number of servers and infinite queues) or busy period distribution. In many cases, it is not possible to obtain these characteristics in exact form and we have to present final results in terms of generating functions or Laplace–Stieltjes transforms. By this way, we are able to calculate at least mean values (or other moments) of the analyzed random variables. Obtained results help us to design telecommunication or computer systems (to choose the proper number of servers, number of waiting positions or to plan parameters connected with customers arrival stream or mean value of their service time) according to given assumptions and taking into account (among others) maximal acceptable loss probability or mean value of waiting time. Moreover, in analyzed models with finite number of servers and finite queues, customer is usually lost only if there are no free servers and no free places in the queue at the epoch of his arrival. So the loss characteristics depend only on the distribution of number of customers present in the system at this epoch, e.g. loss probability P_{loss} in steady-state can be calculated using the obvious formula: $P_{loss} = p_{n+m}$, where n is the number of servers, m is the size of the queue and p_{n+m} is the steady-state probability that there are $n+m$ customers present in the system. In past years, such systems were widely analyzed by many researchers (see e.g. [2, 5, 6, 17, 18]).

It seems to be clear that practical telecommunication or computer systems are rarely composed of servers that can be treated as identical. In many situations, servers can have different service time characteristics. Although we try to obtain similar characteristics for such systems (in this case we can additionally obtain also usage characteristics of all servers), analysis of them is much more complicated because the number of equations describing the system behavior increases exponentially together with increasing number of servers (the number of states in analyzed markovian processes also increases exponentially). In addition, we have to take into account the possible mechanism of choosing the free server (e.g. random choice or fastest server choice). Systems with non-identical (heterogeneous) servers are relatively seldom investigated. The results of these investigations (containing classical queueing models with heterogeneous servers analysis) can be found e.g. in [7, 8, 15–17, 21].

On the other hand, in classical queueing models we assume that customers are homogeneous. In other words, they differ in arriving time moments only, and we do not take into account that they can be different in some other aspects. For example, independently on other customers, a customer can be characterized by some random volume (size) that has an influence on his service time. If we assume the above non-homogenity, we obtain the new class of queueing models containing the systems with customers of random volume. During such investigations we generally have to take into account both possible limitation of the total (customers) volume (which is the sum of the volumes

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of all customers present in the system at some time instant or in steady state) and character of dependency between customer volume and his service time. As summary volume can be limited or unlimited and analyzed random variables (customer volume and his service time) can be dependent or independent, we obtain four types of models. The classification of such models can be found in [18]. The results of the analysis involve not only discussed classical characteristics but also new results like total volume distribution (at any time instant or in steady state). It is worth highlighting in this case the meaning of the loss characteristics for the models with limited total volume. Here we analyze two loss characteristics P_{loss} (part of lost customers) and Q_{loss} (which is the information unit loss probability showing the part of lost information). These characteristics strictly depend on both the memory volume V (that limits total volume) and the character of the dependency between customer volume and his service time. In this case loss probability in the steady state P_{loss} is often bigger than p_{n+m} (the customer volume also has an influence on his possible lost). The above dependency plays very important role – we can design two different systems that are undifferentiated from the classical point of view (e.g. they have the same characteristics of arrival rate and service time) and obtain different final results containing customers number distribution and loss probability for these systems. This is the main reason why such models analysis needed expanding of the previously used classical methods (at the beginning, for analyzing such systems classical queueing models were used but obtained results did not coincide with simulation ones because during the analysis dependency between customer volume and his service time was not exactly taken into account [12, 13]). It is also clear that this approach is more closely related to the real computer (or network) systems in which we have memory limitation and dependence between the service time of the customer (packet) and his volume (size). The practical usage of obtained results is related to the process of designing computer systems in which the customers usually carry some portion of information that is located on servers having limited memory. The above calculations can be exemplary used to choose the proper value of the memory of the computer system according to assumptions connected with maximal acceptable loss characteristics. The importance of such investigations increases together with the progress of computer systems and network solutions (e.g. routers, nodes of telecommunication networks). Some of these models were analyzed e.g. in [18, 22, 23]. The most difficult for analysis are the models with limited total volume and service time depending on customer volume. The exact formulae for steady-state customers number distribution and loss probability were obtained only for systems without waiting positions (Erlang queueing system $M/G/n/(0,V)$ [19] and processor sharing system $M/G/1/(\infty,V) - EPS$ [20]).

In classical queueing theory and its applications to loss networks (that are a collection of resources shared by calls [11]) the Erlang problem plays an important role. The simplest Erlang formula [5, 6] determines the steady-state distribution of number of customers present in the system $M/M/n/0$. Probably the first generalization of this relation was presented by B.A. Sevast'yanov in [14]. In this paper, it was proved that the above distribution depends on the first moment of service time only. So, the Erlang formula was generalized on the system $M/G/n/0$. Further generalizations were concerned with 1) analysis of policy of resource sharing (see e.g. [9, 11]), 2) investigations of systems with non-Poisson customers entrance flow. E.g in [4] the system $BMAP/M/n/0$ was investigated. In this paper, an algorithm of calculation of steady-state distribution of number of customers present in the system and the relation for loss probability were obtained. For all these systems it was assumed that all servers of the system are identical, but, in some of them, customers can belong to different classes dependent on the need of discrete resource (the amount of which is limited in the system) for their service. The next generalization of the Erlang problem concerns with supposition of non-homogenity of customers. It means that each customer is characterized by some random capacity (volume) and the total capacity of customers present in the system is limited by a constant called system capacity (buffer space) V. Such system was analyzed in [19], where it was supposed that customer service time depends on his volume and the amount of the discrete resource that the customer needs during his service. The total amount of this resource in the system is also limited by some constant value.

The aim of present paper is to analyze the $M/\vec{G}/n/(0,V)$ queueing system with non-homogeneous customers, heterogeneous servers and limited (by value V) total volume. This model belongs to the very practical class of models in which we take into account both difference between servers (in practice, all servers are characterized by different service time distributions), non-homogenity of the customers (in many practical applications customers can be understood as packets having some random size and service time of the packet usually depends on its size) and limitation of the total customers volume (e.g. routers have limited memory size). The approach that we use lets calculate exact characteristics of steady-state customers number distribution and loss probability.

The rest of the paper is organized as follows. In the next Section 2, we introduce some necessary notations and random process describing the behavior of the system under consideration together with functions that characterize this process. Section 3 contains an analysis of the mentioned above model. In this section, we obtain the equations for the introduced functions and give their steady-state solution. By this way, we obtain the exact formulae for steady-state customers number distribution and loss probability. In Section 4, we investigate some special cases of the model analyzed in Section 3 and show that the character of the service time and customer volume dependency has an influence on the customers number distribution and loss probability formulae. Finally, Section 5 contains some concluding remarks.

2. The random process and the functions describing the system behavior

Consider the modification of the classical $M/G/n/0$ system in which an arriving customer has additionally some random volume ζ, where ζ is a non-negative random variable. Let $L(x) = P\{\zeta < x\}$ be the distribution function of the random variable ζ. Denote by $\eta(t)$ the number of customers present in the system at time instant t. We assume that service time characteristics can be different for different servers. Denote by $F_j(x,y) = P\{\zeta < x,\ \xi_j < y\},\ j = \overline{1,n}$, the joint distribution function of the customer volume and his service time for j-th server. Assume that the total customers volume $\sigma(t)$, which is the sum of the volumes of all customers present in the system at time instant t, is limited by the value V, $V > 0$. It means that the arriving at epoch t customer whose volume is equal to x will be accepted to the service if $\eta(t^-) < n$ and $\sigma(t^-) + x \le V$. In opposite case, he will be lost without any influence on the future system behavior. We also assume that a customer accepted to service chooses one of free servers randomly. It means that he will be served by one of l free servers with probability $1/l$.

For the system under consideration, we shall obtain the formulae for the steady-state customers number distribution and loss probability. To do this, we will use the generalized method of auxiliary variables [2]. We denote by a the parameter of customers entrance flow and by $B_j(y)$ the distribution function of service time ξ_j for j-th server. Let β_j be the mean value of the random variable ξ_j. It is obvious that $L(x) = F_j(x,\infty)$ for all $j = \overline{1,n}$ and $B_j(y) = F_j(\infty,y)$. We also assume that all servers are numbered by the natural numbers from 1 to n. Let $A(t)$ be the set of the numbers of busy servers in the system at time instant t. We denote the rest of the service time of the customer (that is served by j-th server at time instant t) as $\xi_j^*(t)$ and the volumes of the present customers as $\chi_j(t)$, $j \in A(t)$. It is clear that $\sigma(t) = \sum_{j\in A(t)} \chi_j(t)$.

We shall characterize the system behavior by the following markovian process:

$$\left(\eta(t), A(t), \xi_j^*(t), \chi_j(t),\ \ j \in A(t)\right). \tag{1}$$

Later on, we shall denote the analyzed system by the notation $M/\vec{G}/n/(0,V)$. It is clear that, in the case of empty system ($\eta(t) = 0$), the process (1) reduces to $\eta(t)$.

The process (1) is characterized by the following functions (see [23]):

$$P_k(t) = P\{\eta(t) = k\}, \quad k = \overline{0,n}; \tag{2}$$

$$P_k^{\{i_1,\dots,i_k\}}(t) = P\{\eta(t) = k,\ A(t) = \{i_1,\dots,i_k\}\},\quad k = \overline{1,n}; \tag{3}$$

$$G_k^{\{i_1,\dots,i_k\}}(y_1,\dots,y_k,t) = P\{\eta(t) = k,\ A(t) = \{i_1,\dots,i_k\},\ \xi_{i_j}^*(t) < y_j,\ j = \overline{1,k}\},\quad k = \overline{1,n}; \tag{4}$$

$$H_k^{\{i_1,\dots,i_k\}}(x,y_1,\dots,y_k,t) = P\{\eta(t)=k,\ A(t)=\{i_1,\dots,i_k\},\ \sigma(t) < x,\ \xi_{i_j}^*(t) < y_j,\ j = \overline{1,k}\},\quad k = \overline{1,n}. \tag{5}$$

If $k = n$, the functions (4) and (5) are denoted by $G_n(y_1,\dots,y_n,t)$ and $H_n(x,y_1,\dots,y_n,t)$, respectively.

In steady-state (if the inequality $a\beta_j < \infty$, $j = \overline{1,n}$, holds), we can introduce the stationary analogies of the functions (2)–(5):

$$p_k = \lim_{t\to\infty} P_k(t), \quad k = \overline{0,n}; \tag{6}$$

$$p_k^{\{i_1,\dots,i_k\}} = \lim_{t\to\infty} P_k^{\{i_1,\dots,i_k\}}(t), \quad k = \overline{1,n}; \tag{7}$$

$$g_k^{\{i_1,\dots,i_k\}}(y_1,\dots,y_k) = \lim_{t\to\infty} G_k^{\{i_1,\dots,i_k\}}(y_1,\dots,y_k,t), \quad k = \overline{1,n}; \tag{8}$$

$$h_k^{\{i_1,\dots,i_k\}}(x,y_1,\dots,y_k) = \lim_{t\to\infty} H_k^{\{i_1,\dots,i_k\}}(x,y_1,\dots,y_k,t), \quad k = \overline{1,n}. \tag{9}$$

If $k = n$, we use the notations $g_n(y_1,\dots,y_n)$ and $h_n(x,y_1,\dots,y_n)$ instead of $g_n^{\{i_1,\dots,i_n\}}(y_1,\dots,y_n)$ and $h_n^{\{i_1,\dots,i_n\}}(x,y_1,\dots,y_n)$, respectively. Note that the functions (4), (5) and (8), (9) are not symmetrical in regard to permutations of the variables y_j, $j = \overline{1,k}$, because the servers are not identical in the system under consideration.

It is clear that

$$p_k^{\{i_1,\dots,i_k\}} = g_k^{\{i_1,\dots,i_k\}}(\infty,\dots,\infty), \tag{10}$$

$$g_k^{\{i_1,\dots,i_k\}}(y_1,\dots,y_k) = h_k^{\{i_1,\dots,i_k\}}(V,y_1,\dots,y_k), \quad k=\overline{1,n}. \tag{11}$$

3. An analysis of the model

First, we analyze, for simplicity, the system with two different servers ($n = 2$). In this case, we can write down the following difference equations:

$$P_0(t+\Delta t) = P_0(t)(1 - a\Delta t L(V)) + G_1^{\{1\}}(\Delta t,t) + G_1^{\{2\}}(\Delta t,t) + o(\Delta t); \tag{12}$$

$$G_1^{\{1\}}(y,t+\Delta t) = \frac{a\Delta t}{2} P_0(t) F_1(V,y+\Delta t) + G_1^{\{1\}}(y+\Delta t,t) - G_1^{\{1\}}(\Delta t,t) - a\Delta t \int_0^V H_1^{\{1\}}(V-x,y+\Delta t,t)\,\mathrm{d}L(x) + G_2(y+\Delta t,\Delta t,t) + o(\Delta t); \tag{13}$$

$$G_1^{\{2\}}(y,t+\Delta t) = \frac{a\Delta t}{2} P_0(t) F_2(V,y+\Delta t) + G_1^{\{2\}}(y+\Delta t,t) - G_1^{\{2\}}(\Delta t,t) - a\Delta t \int_0^V H_1^{\{2\}}(V-x,y+\Delta t,t)\,\mathrm{d}L(x) + G_2(\Delta t,y+\Delta t,t) + o(\Delta t); \tag{14}$$

$$\begin{aligned} G_2(y_1,y_2,t+\Delta t) = {} & a\Delta t \int_0^V \Big(H_1^{\{1\}}(V-x,y_1+\Delta t,t) - H_1^{\{1\}}(V-x,\Delta t,t)\Big)\,\mathrm{d}_x F_2(x,y_2+\Delta t) \\ & + a\Delta t \int_0^V \Big(H_1^{\{2\}}(V-x,y_2+\Delta t,t) - H_1^{\{2\}}(V-x,\Delta t,t)\Big)\,\mathrm{d}_x F_1(x,y_1+\Delta t) \\ & + G_2(y_1+\Delta t,y_2+\Delta t,t) - G_2(\Delta t,y_2+\Delta t,t) \\ & - G_2(y_1+\Delta t,\Delta t,t) + o(\Delta t). \end{aligned} \tag{15}$$

If $\Delta t \to 0$, then, from the equations (12)–(15), we obtain the following system of the partial differential equations:

$$\frac{\mathrm{d}P_0(t)}{\mathrm{d}t} = -aP_0(t)L(V) + \left.\frac{\partial G_1^{\{1\}}(y,t)}{\partial y}\right|_{y=0} + \left.\frac{\partial G_1^{\{2\}}(y,t)}{\partial y}\right|_{y=0}; \quad (16)$$

$$\frac{\partial G_1^{\{1\}}(y,t)}{\partial t} - \frac{\partial G_1^{\{1\}}(y,t)}{\partial y} + \left.\frac{\partial G_1^{\{1\}}(y,t)}{\partial y}\right|_{y=0} = \frac{a}{2}P_0(t)F_1(V,y) - a\int_0^V H_1^{\{1\}}(V-x,y,t)\,\mathrm{d}L(x) + \left.\frac{\partial G_2(y,u,t)}{\partial u}\right|_{u=0}; \quad (17)$$

$$\frac{\partial G_1^{\{2\}}(y,t)}{\partial t} - \frac{\partial G_1^{\{2\}}(y,t)}{\partial y} + \left.\frac{\partial G_1^{\{2\}}(y,t)}{\partial y}\right|_{y=0} = \frac{a}{2}P_0(t)F_2(V,y) - a\int_0^V H_1^{\{2\}}(V-x,y,t)\,\mathrm{d}L(x) + \left.\frac{\partial G_2(u,y,t)}{\partial u}\right|_{u=0}; \quad (18)$$

$$\frac{\partial G_2(y_1,y_2,t)}{\partial t} - \frac{\partial G_2(y_1,y_2,t)}{\partial y_1} - \frac{\partial G_2(y_1,y_2,t)}{\partial y_2} + \left.\frac{\partial G_2(y_1,y_2,t)}{\partial y_1}\right|_{y_1=0} + \left.\frac{\partial G_2(y_1,y_2,t)}{\partial y_2}\right|_{y_2=0} = a\int_0^V H_1^{\{1\}}(V-x,y_1,t)\,\mathrm{d}_xF_2(x,y_2) + a\int_0^V H_1^{\{2\}}(V-x,y_2,t)\,\mathrm{d}_xF_1(x,y_1). \quad (19)$$

In steady state ($t \to \infty$), we easily obtain from (16)–(19):

$$0 = -ap_0L(V) + \left.\frac{\partial g_1^{\{1\}}(y)}{\partial y}\right|_{y=0} + \left.\frac{\partial g_1^{\{2\}}(y)}{\partial y}\right|_{y=0}; \quad (20)$$

$$-\frac{\partial g_1^{\{1\}}(y)}{\partial y} + \left.\frac{\partial g_1^{\{1\}}(y)}{\partial y}\right|_{y=0} = \frac{a}{2}p_0F_1(V,y) - a\int_0^V h_1^{\{1\}}(V-x,y)\,\mathrm{d}L(x) + \left.\frac{\partial g_2(y,u)}{\partial u}\right|_{u=0}; \quad (21)$$

$$-\frac{\partial g_1^{\{2\}}(y)}{\partial y} + \left.\frac{\partial g_1^{\{2\}}(y)}{\partial y}\right|_{y=0} = \frac{a}{2}p_0F_2(V,y) - a\int_0^V h_1^{\{2\}}(V-x,y)\,\mathrm{d}L(x) + \left.\frac{\partial g_2(u,y)}{\partial u}\right|_{u=0}; \quad (22)$$

$$-\frac{\partial g_2(y_1,y_2)}{\partial y_1} - \frac{\partial g_2(y_1,y_2)}{\partial y_2} + \left.\frac{\partial g_2(y_1,y_2)}{\partial y_1}\right|_{y_1=0} + \left.\frac{\partial g_2(y_1,y_2)}{\partial y_2}\right|_{y_2=0} = a\int_0^V h_1^{\{1\}}(V-x,y_1)d_xF_2(x,y_2) + a\int_0^V h_1^{\{2\}}(V-x,y_2)d_xF_1(x,y_1). \quad (23)$$

In steady state, the following boundary conditions take place:

$$\left.\frac{\partial g_2(y_1,y_2)}{\partial y_1}\right|_{y_1=0} = a\int_0^V h_1^{\{2\}}(V-x,y_2)\,\mathrm{d}L(x); \quad (24)$$

$$\left.\frac{\partial g_2(y_1,y_2)}{\partial y_2}\right|_{y_2=0} = a\int_0^V h_1^{\{1\}}(V-x,y_1)\,\mathrm{d}L(x). \quad (25)$$

The normalization condition has the following form:

$$p_0 + g_1^{\{1\}}(\infty) + g_1^{\{2\}}(\infty) + g_2(\infty,\infty) = 1. \quad (26)$$

To write out the solution of the equations (20)–(23) with the boundary conditions (24), (25), we need some subsidary notations. Denote by $K_j(x,y)$ the probability than an arbitrary customer that is served by j-th server ($j = 1,2$) has a volume less than x and service time greater or equal y i.e.

$$\begin{aligned} K_j(x,y) &= P\{\zeta < x,\ \xi_j \geq y\} \\ &= P\{\zeta < x\} - P\{\zeta < x,\ \xi_j < y\} \\ &= L(x) - F_j(x,y), \quad j = 1,2. \end{aligned} \quad (27)$$

Introduce one more function that is defined as follows:

$$R_j^y(x) = \int_0^y K_j(x,u)\,\mathrm{d}u, \quad j = 1,2. \quad (28)$$

We also use the following notation for the Stieltjes convolution of the functions $F_1(x)$ and $F_2(x)$:

$$F_1 * F_2(x) = \int_0^x F_1(x-u)\,\mathrm{d}F_2(u). \quad (29)$$

By direct substitution, taking into account the boundary conditions (24), (25), we can check that the solution of the equations (20)–(23) has the following form:

$$h_1^{\{j\}}(x,y) = \frac{ap_0}{2}R_j^y(x), \quad j = 1,2; \quad (30)$$

$$g_1^{\{j\}}(y) = \frac{ap_0}{2}R_j^y(V), \quad j = 1,2; \quad (31)$$

$$g_2(y_1,y_2) = \frac{a^2p_0}{2}R_1^{y_1} * R_2^{y_2}(V). \quad (32)$$

Now we introduce one more notation:

$$D_j(x) = \lim_{y\to\infty} R_j^y(x), \quad j = 1,2. \tag{33}$$

Taking into consideration relation (10), we finally obtain:

$$p_1^{\{j\}} = \frac{ap_0}{2} D_j(V), \quad j = 1,2; \tag{34}$$

$$p_1 = p_1^{\{1\}} + p_1^{\{2\}} = \frac{ap_0}{2}\left(D_1(V) + D_2(V)\right); \tag{35}$$

$$p_2 = \frac{a^2 p_0}{2} D_1 * D_2(V) \tag{36}$$

and

$$p_0 = \left[1 + \frac{a}{2}(D_1(V) + D_2(V)) + \frac{a^2}{2} D_1 * D_2(V)\right]^{-1}, \tag{37}$$

as it follows from the relation (26).

We can analyze the general case of $M/\vec{G}/n/(0,V)$ system with heterogeneous servers, non-homogeneous customers and limited memory space by analogous way. It is clear that the number of equations describing the system behavior increases exponentially together with increasing number of servers n. For simplicity, we introduce the following notations. Let $\{C_k^n\}$ be the set of all k-element combinations of the n-element set. Then, the steady-state equations for the system under consideration can be presented in the following form:

$$0 = -ap_0 L(V) + \sum_{i=1}^{n} \left.\frac{\partial g_1^{\{i\}}(y)}{\partial y}\right|_{y=0}; \tag{38}$$

$$\begin{aligned} -\frac{\partial g_1^{\{i\}}(y)}{\partial y} + \left.\frac{\partial g_1^{\{i\}}(y)}{\partial y}\right|_{y=0} &= \frac{a}{n} p_0 F_i(V,y) \\ &- a\int_0^V h_1^{\{i\}}(V-x,y)\,\mathrm{d}L(x) + \sum_{j=1}^{i-1} \left.\frac{\partial g_2^{\{i,j\}}(u,y)}{\partial u}\right|_{u=0} \\ &+ \sum_{j=i+1}^{n} \left.\frac{\partial g_2^{\{i,j\}}(y,u)}{\partial u}\right|_{u=0}, \quad i = \overline{1,n}; \end{aligned} \tag{39}$$

$$\begin{aligned} -\sum_{i=1}^{k} \frac{\partial g_k^{\{i_1,\dots,i_k\}}(y_1,\dots,y_k)}{\partial y_i} &+ \sum_{i=1}^{k} \left.\frac{\partial g_k^{\{i_1,\dots,i_k\}}(y_1,\dots,y_k)}{\partial y_i}\right|_{y_i=0} \\ = \frac{a}{n-k+1} \sum_{j=1}^{k} \int_0^V & h_{k-1}^{\{i_1,\dots,i_{j-1},i_{j+1},\dots,i_k\}}(V-x,y_1,\dots,y_{j-1},y_{j+1},\dots,y_k) \\ \times\, \mathrm{d}_x F_j(x,y_j) &- a\int_0^V h_k^{\{i_1,\dots,i_k\}}(V-x,y_1,\dots,y_k)\,\mathrm{d}L(x) \\ + \sum_{j\notin\{i_1,\dots,i_k\}} & \left.\frac{\partial g_{k+1}^{\{i_1,\dots,i_k,j\}}(y_1,\dots,y_{j-1},u,y_j,\dots,y_k)}{\partial u}\right|_{u=0}, \\ \{i_1,\dots,i_k\} &\in \{C_k^n\}, \quad k = \overline{2,n-1}; \end{aligned} \tag{40}$$

$$\begin{aligned} -\sum_{i=1}^{n} \frac{\partial g_n(y_1,\dots,y_n)}{\partial y_i} &+ \sum_{i=1}^{n} \left.\frac{\partial g_n(y_1,\dots,y_n)}{\partial y_i}\right|_{y_i=0} \\ = a\sum_{j=1}^{n} \int_0^V & h_{n-1}^{\{i_1,\dots,i_{j-1},i_{j+1},\dots,i_n\}}(V-x,y_1,\dots,y_{j-1},y_{j+1},\dots,y_n) \\ & \times\, \mathrm{d}_x F_j(x,y_j). \end{aligned} \tag{41}$$

The following boundary conditions take place:

$$\begin{aligned} \left.\frac{\partial g_k^{\{i_1,\dots,i_k\}}(y_1,\dots,y_k)}{\partial y_j}\right|_{y_j=0} &= \frac{a}{n-k+1} \int_0^V h_{k-1}^{\{i_1,\dots,i_{j-1},i_{j+1},\dots,i_k\}}(V-x,y_j)\,\mathrm{d}L(x), \\ j &= \overline{1,k}, \quad k = \overline{2,n}. \end{aligned} \tag{42}$$

By direct substitution, taking into account the boundary conditions (42) and using analogously defined functions $R_j^y(x)$ and $D_j(x)$, we can check that the solution of equations (38)–(41) has the following form:

$$h_1^{\{j\}}(x,y) = \frac{ap_0}{n} R_j^y(x), \quad j = \overline{1,n}; \tag{43}$$

$$\begin{aligned} h_k^{\{i_1,\dots,i_k\}}(x,y_1,\dots,y_k) &= \frac{a^k(n-k)!\,p_0}{n!} R_{i_1}^{y_1} * \dots * R_{i_k}^{y_k}(x), \\ \{i_1,\dots,i_k\} &\in \{C_k^n\}, \quad k = \overline{2,n-1}; \end{aligned} \tag{44}$$

$$g_1^{\{j\}}(y) = \frac{ap_0}{n} R_j^y(V), \quad j = \overline{1,n}; \tag{45}$$

$$\begin{aligned} g_k^{\{i_1,\dots,i_k\}}(y_1,\dots,y_k) &= \frac{a^k(n-k)!\,p_0}{n!} R_{i_1}^{y_1} * \dots * R_{i_k}^{y_k}(V), \\ \{i_1,\dots,i_k\} &\in \{C_k^n\}, \quad k = \overline{2,n}; \end{aligned} \tag{46}$$

$$p_1^{\{j\}} = \frac{ap_0}{n} D_j(V), \quad j = \overline{1,n}; \tag{47}$$

$$\begin{aligned} p_k^{\{i_1,\dots,i_k\}} &= \frac{a^k(n-k)!\,p_0}{n!} D_{i_1} * \dots * D_{i_k}(V), \\ \{i_1,\dots,i_k\} &\in \{C_k^n\}, \quad k = \overline{2,n}. \end{aligned} \tag{48}$$

Finally, we obtain steady-state customers number distribution in the following form:

$$p_1 = \frac{ap_0}{n} \sum_{j=1}^{n} D_j(V); \tag{49}$$

$$p_k = \frac{a^k(n-k)!\,p_0}{n!} \sum_{\{i_1,\dots,i_k\}\in\{C_k^n\}} D_{i_1} * \dots * D_{i_k}(V), \tag{50}$$

$$k = \overline{2,n},$$

where

$$p_0 = \left[1 + \frac{a}{n}\sum_{j=1}^{n} D_j(V) + \frac{1}{n!}\sum_{k=2}^{n} a^k (n-k)! \sum_{\{i_1,\ldots,i_k\}\in\{C_k^n\}} D_{i_1} * \ldots * D_{i_k}(V)\right]^{-1} \quad (51)$$

can be obtained from the normalization condition $\sum_{k=0}^{n} p_k = 1$.

As it follows from the obtained relations, the steady-state customers number distribution in the system under consideration depends not only on the mean values (β_j) of service time (as it takes place in the classical $M/\vec{G}/n/0$ model [23]) but it also depends on the joint distribution fuction of the customer volume and his service time. Formulae (47)–(51) can be used for calculating of some very practical characteristics, including the mean value of the steady-state number of customers present in the system $(E\eta = \sum_{k=0}^{n} k p_k)$ and steady-state distribution of usage of j-th server Q_j $(j = \overline{1,n})$. Indeed, we have

$$Q_j = \sum_{k=1}^{n} P(\{C_{k,j}^n\}), \quad (52)$$

where $\{C_{k,j}^n\}$ denotes the set of all k-element subsets of the set of the numbers of servers containing j and $P(\{C_{k,j}^n\})$ denotes the probability that we find our system in a state from the set $\{C_{k,j}^n\}$ e.g. for the system $M/\vec{G}/3/(0,V)$ we obtain:

$$Q_1 = p_1^{\{1\}} + p_2^{\{1,2\}} + p_2^{\{1,3\}} + p_3\,,$$
$$Q_2 = p_1^{\{2\}} + p_2^{\{1,2\}} + p_2^{\{2,3\}} + p_3\,,$$
$$Q_3 = p_1^{\{3\}} + p_2^{\{1,3\}} + p_2^{\{2,3\}} + p_3\,.$$

Now we find a formula for the steady-state loss probability. It is clear that it is not equal to p_n as it takes place for the classical $M/\vec{G}/n/0$ model [23]. The arriving customer can be lost not only in the case when there are no free servers at his arriving epoch, but also if his volume is too big, for the reason that the total volume is limited.

To find the formula for the steady-state loss probability P_{loss}, we use the equilibrium condition. This condition is based on the fact that, in steady state, the average number of customers accepted for service within a time unit is equal to the average number of customers whose service was terminated within this time period. Therefore, we obtain the following equation:

$$a(1-P_{loss}) = \sum_{k=1}^{n} \sum_{\{i_1,\ldots,i_k\}\in\{C_k^n\}} \sum_{j=1}^{k} \frac{\partial g_k^{\{i_1,\ldots,i_k\}}(\infty_{j-1}, z, \infty_{k-j})}{\partial z}\bigg|_{z=0}, \quad (53)$$

where $\infty_j = (\infty,\ldots,\infty)$ denotes the vector with j components, from which we have:

$$P_{loss} = 1 - \frac{1}{a} \times \left[\sum_{k=1}^{n} \sum_{\{i_1,\ldots,i_k\}\in\{C_k^n\}} \sum_{j=1}^{k} \frac{\partial g_k^{\{i_1,\ldots,i_k\}}(\infty_{j-1}, z, \infty_{k-j})}{\partial z}\bigg|_{z=0}\right]. \quad (54)$$

Taking into account the relations (45), (46), we easily obtain:

$$\frac{\partial g_1^{\{j\}}(z)}{\partial z}\bigg|_{z=0} = \frac{a p_0}{n} L(V), \quad j = \overline{1,n}; \quad (55)$$

$$\frac{\partial g_k^{\{i_1,\ldots,i_k\}}(\infty_{j-1}, z, \infty_{k-j})}{\partial z}\bigg|_{z=0} = \frac{a^k p_0 (n-k)!}{n!} \times D_{i_1} * \ldots * D_{i_{j-1}} * L * D_{i_{j+1}} * \ldots * D_{i_k}(V), \quad \{i_1,\ldots,i_k\} \in \{C_k^n\}, \quad k = \overline{2,n}. \quad (56)$$

Finally, the loss probability formula takes the form:

$$P_{loss} = 1 - p_0 \left[L(V) + \frac{1}{n!}\sum_{k=2}^{n} a^{k-1}(n-k)! \times \sum_{\{i_1,\ldots,i_k\}\in\{C_k^n\}} \sum_{j=1}^{k} D_{i_1} * \ldots * D_{i_{j-1}} * L * D_{i_{j+1}} * \ldots * D_{i_k}(V)\right]. \quad (57)$$

4. Special cases and numerical examples

In this section, we consider two practical special cases of the analyzed model. In the first case, customer volume and his service time on j-th server, $j = \overline{1,n}$, are independent. The second case presents a situation in which service time on j-th server is proportional to customer volume with coefficient c_j i.e. $\xi_j = c_j \zeta$. For these special cases, we obtain the formulae for the steady-state customers number distribution and loss probabilty and present some numerical results.

4.1. Service time and customer volume are independent. Assume that customer volume and his service time are independent for every server. It is obvious that $F_j(x,y) = L(x)B_j(y)$ in this case. Then, the formulae (45)–(51) take the following form:

$$g_1^{\{j\}}(y) = \frac{a p_0 L(V)}{n} \int_0^y [1 - B_j(u)]\,\mathrm{d}u, \quad j = \overline{1,n}; \quad (58)$$

$$g_k^{\{i_1,\ldots,i_k\}}(y_1,\ldots,y_k) = \frac{a^k (n-k)! p_0}{n!} \prod_{j=1}^{k} \int_0^{y_j} [1 - B_{i_j}(u)]\,\mathrm{d}u\, L_k^*(V), \quad \{i_1,\ldots,i_k\} \in \{C_k^n\}, \quad k = \overline{2,n}; \quad (59)$$

$$p_1^{\{j\}} = \frac{a p_0 L(V) \beta_j}{n}, \quad j = \overline{1,n}; \quad (60)$$

$$p_k^{\{i_1,\dots,i_k\}} = \frac{a^k(n-k)!p_0}{n!}\prod_{j=1}^{k}\beta_{i_j}\,L_k^*(V),\quad \{i_1,\dots,i_k\}\in\{C_k^n\},\quad k=\overline{2,n}; \tag{61}$$

$$p_1 = \frac{ap_0L(V)}{n}\sum_{j=1}^{n}\beta_j; \tag{62}$$

$$p_k = \frac{a^k(n-k)!p_0}{n!}\sum_{\{i_1,\dots,i_k\}\in\{C_k^n\}}\prod_{j=1}^{k}\beta_{i_j}\,L_k^*(V),\quad k=\overline{2,n}; \tag{63}$$

$$p_0 = \left[1+\frac{aL(V)}{n}\sum_{j=1}^{n}\beta_j + \frac{1}{n!}\sum_{k=2}^{n}a^k(n-k)!\sum_{\{i_1,\dots,i_k\}\in\{C_k^n\}}\prod_{j=1}^{k}\beta_{i_j}\,L_k^*(V)\right]^{-1}, \tag{64}$$

where $L_k^*(x)$ denotes the k-fold Stieltjes convolution of the function $L(x)$.

Loss probability formula takes the form:

$$P_{loss} = 1-p_0\left[L(V)+\frac{1}{n!}\sum_{k=2}^{n}a^{k-1}(n-k)! \times \sum_{\{i_1,\dots,i_k\}\in\{C_k^n\}}\sum_{j=1}^{k}\prod_{l=1,\ l\neq j}^{k}\beta_{i_l}\,L_k^*(V)\right]. \tag{65}$$

It is clear that we can use the above formulae to calculate exact characteristics only if the convolutions $L_k^*(x)$ can be obtained in exact form. In other cases, we can use approximating methods.

Let us additionally assume that customer volume is exponentially distributed with parameter f and his service time on each of n servers is exponentially distributed with parameters $\mu_1,\dots,\mu_n$, respectively. Then, we obtain the following exact formulae:

$$p_1 = \frac{ap_0\left(1-e^{-fV}\right)}{n}\sum_{j=1}^{n}\frac{1}{\mu_j}; \tag{66}$$

$$p_k = \frac{a^k(n-k)!p_0}{n!}\sum_{\{i_1,\dots,i_k\}\in\{C_k^n\}}\prod_{j=1}^{k}\frac{1}{\mu_{i_j}}\left(1-e^{-fV}\sum_{l=0}^{k-1}\frac{(fV)^l}{l!}\right),\quad k=\overline{2,n}; \tag{67}$$

$$p_0 = \left[1+\frac{a\left(1-e^{-fV}\right)}{n}\sum_{j=1}^{n}\frac{1}{\mu_j} + \frac{1}{n!}\sum_{k=2}^{n}a^k(n-k)!\sum_{\{i_1,\dots,i_k\}\in\{C_k^n\}}\prod_{j=1}^{k}\frac{1}{\mu_{i_j}}\left(1 - e^{-fV}\sum_{l=0}^{k-1}\frac{(fV)^l}{l!}\right)\right]^{-1}; \tag{68}$$

$$P_{loss} = 1-p_0\left[1-e^{-fV}+\frac{1}{n!}\sum_{k=2}^{n}a^{k-1}(n-k)! \times \sum_{\{i_1,\dots,i_k\}\in\{C_k^n\}}\sum_{j=1}^{k}\prod_{l=1,\ l\neq j}^{k}\frac{1}{\mu_{i_l}}\left(1-e^{-fV}\sum_{l=0}^{k-1}\frac{(fV)^l}{l!}\right)\right]. \tag{69}$$

Now, we illustrate obtained theoretical results with some numerical example. Let us consider $M/\vec{M}/3/(0,V)$ queueing system in which service times are exponentially distributed with parameters $\mu_1=2$, $\mu_2=4$ and $\mu_3=5$. Service times and customer volumes are independent and the customer volume is exponentially distributed with parameter $f=1$. The parameter of a Poisson entrance flow is equal to $a=1$. Then, using the relations (66)–(69), we obtain the following results:

$$p_1 = \frac{19}{60}p_0\left(1-e^{-V}\right);$$

$$p_2 = \frac{11}{240}p_0\left(1-e^{-V}(1+V)\right);$$

$$p_3 = \frac{1}{240}p_0\left(1-e^{-V}\left(1+V+\frac{V^2}{2}\right)\right);$$

$$p_0 = \left[1+\frac{19}{60}\left(1-e^{-V}\right)+\frac{11}{240}\left(1-e^{-V}(1+V)\right) + \frac{1}{240}\left(1-e^{-V}\left(1+V+\frac{V^2}{2}\right)\right)\right]^{-1}; \tag{70}$$

$$P_{loss} = 1-p_0\left[1-e^{-V}+\frac{19}{60}\left(1-e^{-V}(1+V)\right) + \frac{11}{240}\left(1-e^{-V}\left(1+V+\frac{V^2}{2}\right)\right)\right].$$

Now we present obtained results in two tables. Table 1 presents the results for the steady-state customers number distribution p_k, $k=\overline{0,3}$ and loss probability P_{loss} obtained using (70). Calculations were done using previously prepared Python3 scripts [3], whereas in Table 2 we present the same results obtained by simulation. Simulation was done with the usage of discrete event simulation method (DES) that is discussed e.g. in [10] with the help of previously prepared Python3 programs [3]. As the moments of discrete events in the above method (generated with the usage of computer random generators) we take here the moments when state of the system changes (in whiles of customers arrival or their service termination). During simulation we collect (sum) all time characteristics of system presence in state k (k customers present in the system). We use obtained statistics T_k to calculate this part of simulation time TM in which system is in state k. This p_k^{SIM} statistics equals T_k/TM and converges to p_k if $TM\to\infty$ (we choose enough long simulation time TM). By analogous way, we also calculate this part of the number of arriving customers that are lost (this P_{loss}^{SIM} statistics converges to P_{loss}). We can see that the simulation results (Table 2) confirm theoretical ones (Table 1).

In the next numerical example, we investigate $M/\vec{G}/2/(0,V)$ system with Poisson entrance flow parameter $a=4$. Assume

Table 1
Customers number distribution and loss probability for $M/\vec{M}/3/(0,V)$ system – service times and customer volumes are independent – theoretical results

V	p_0	p_1	p_2	p_3	P_{loss}
1	0.8247	0.1651	0.0100	0.0003	0.4067
2	0.7678	0.2102	0.0209	0.0010	0.1803
3	0.7463	0.2246	0.0274	0.0018	0.0819
4	0.7376	0.2293	0.0307	0.0023	0.0379
5	0.7341	0.2309	0.0323	0.0027	0.0183
6	0.7327	0.2314	0.0330	0.0029	0.0096
7	0.7321	0.2316	0.0333	0.0030	0.0059
8	0.7319	0.2317	0.0334	0.0030	0.0042
9	0.7318	0.2317	0.0335	0.0030	0.0035
10	0.7317	0.2317	0.0335	0.0030	0.0033

Table 2
Customers number distribution and loss probability for $M/\vec{M}/3/(0,V)$ system – service times and customer volumes are independent – simulation results

V	p_0^{SIM}	p_1^{SIM}	p_2^{SIM}	p_3^{SIM}	P_{loss}^{SIM}
1	0.8245	0.1652	0.0100	0.0003	0.4065
2	0.7677	0.2103	0.0209	0.0010	0.1806
3	0.7468	0.2241	0.0273	0.0018	0.0819
4	0.7379	0.2290	0.0308	0.0023	0.0378
5	0.7350	0.2302	0.0321	0.0027	0.0182
6	0.7327	0.2314	0.0330	0.0029	0.0095
7	0.7327	0.2312	0.0331	0.0030	0.0058
8	0.7327	0.2311	0.0332	0.0030	0.0043
9	0.7314	0.2321	0.0335	0.0030	0.0036
10	0.7325	0.2312	0.0334	0.0030	0.0032

that volumes of the arriving customers are uniformly distributed on the interval $[0,2]$ and service times on each of two servers are also uniformly distributed on the intervals $[0,5]$ and $[0,10]$, respectively and do not depend on the customer volume. For analyzed example, obtaining the exact form of convolutions appearing in (64–65) is more complicated and we use the following method of calculating the convolution $F_1 * F_2(x)$: we calculate the Laplace-Stieltjes transforms (LSTs) $\alpha_1(q)$ and $\alpha_2(q)$ of the functions $F_1(x)$ and $F_2(x)$, respectively. So the LST of the convolution $F_1 * F_2(x)$ has the form $\alpha(q) = \alpha_1(q)\alpha_2(q)$. It is clear that function $\gamma(q) = \dfrac{\alpha(q)}{q}$ is the Laplace transform of the above convolution. Finally, we use Laplace transform inversion to obtain the formula for $F_1 * F_2(x)$. This inversion is possible to calculate with the help of *Mathematica* environment [1]. The discussed method is widely investigated e.g. in [25]. Thus, basing on (62)–(65) formulae, after some calculations, we obtain the following results:

$$
\begin{aligned}
p_1 &= \frac{15}{2} p_0 \left[V - (V-2)H(V-2)\right];\\
p_2 &= \frac{25}{2} p_0 \left[(V-4)^2 H(V-4) - 2(V-2)^2 H(V-2) + V^2\right];\\
p_0 &= \Bigg\{1 + \frac{15}{2}\left[V - (V-2)H(V-2)\right]\\
&\quad + \frac{25}{2}\left[(V-4)^2 H(V-4) - 2(V-2)^2 H(V-2) + V^2\right]\Bigg\}^{-1};\\
P_{loss} &= 1 - p_0 \Bigg\{\frac{1}{2}V - \frac{1}{2}(V-2)H(V-2)\\
&\quad + \frac{15}{8}\left[(V-4)^2 H(V-4) - 2(V-2)^2 H(V-2) + V^2\right]\Bigg\},
\end{aligned}
\tag{71}
$$

where $H(x)$ denotes the Heaviside unit step fuction.

In Table 3 and Table 4 we present the results obtained using (71) and DES simulation method, respectively. It is clear that,

Table 3
Customers number distribution and loss probability for $M/\vec{G}/2/(0,V)$ system – service times and customer volumes are independent (uniform distributions) – theoretical results

V	p_0	p_1	p_2	P_{loss}
0.5	0.1270	0.4762	0.3968	0.9087
1.0	0.0476	0.3571	0.5952	0.8869
1.5	0.0248	0.2786	0.6966	0.8769
2.0	0.0152	0.2273	0.7576	0.8712
2.5	0.0114	0.1707	0.8179	0.8659
3.0	0.0097	0.1449	0.8454	0.8635
3.5	0.0089	0.1329	0.8583	0.8624
4.0	0.0086	0.1293	0.8621	0.8621

Table 4
Customers number distribution and loss probability for $M/\vec{G}/2/(0,V)$ system – service times and customer volumes are independent (uniform distributions) – simulation results

V	p_0^{SIM}	p_1^{SIM}	p_2^{SIM}	P_{loss}^{SIM}
0.5	0.1273	0.4761	0.3966	0.9089
1.0	0.0477	0.3575	0.5948	0.8870
1.5	0.0247	0.2785	0.6967	0.8769
2.0	0.0151	0.2272	0.7578	0.8712
2.5	0.0114	0.1705	0.8181	0.8658
3.0	0.0095	0.1450	0.8455	0.8635
3.5	0.0089	0.1335	0.8576	0.8622
4.0	0.0086	0.1294	0.8620	0.8620

in analyzed case, $P_{loss} \to p_2$ if $V \to 4$ because if $V \geq 4$, then the volume of the arriving customer has no influence on his possible lost (total volume of two customers is less or equal 4).

4.2. Service time is proportional to customer volume. Assume that the service time on j-th server is proportional to the customer volume with coefficient c_j i.e. $\xi_j = c_j\zeta, j = \overline{1,n}$. In this case, we obtain the obvious formula:

$$F_j(x,y) = P\{\zeta < x,\ \xi_j < y\} = P\{\zeta < x,\ c_j\zeta < y\} = P\left\{\zeta < x,\ \zeta < \frac{y}{c_j}\right\} = L\left(\min\left(x, \frac{y}{c_j}\right)\right). \tag{72}$$

Then, the formulae (45)–(46) take the following form:

$$g_1^{\{j\}}(y) = \frac{ap_0}{n}\left[L(V)y - \int_0^y L\left(\min\left(V, \frac{u}{c_j}\right)\right)\mathrm{d}u\right], \quad j = \overline{1,n}; \tag{73}$$

$$g_k^{\{i_1,\ldots,i_k\}}(y_1,\ldots,y_k) = \frac{a^k(n-k)!p_0}{n!}R_{i_1}^{y_1} * \ldots * R_{i_k}^{y_k}(V), \quad \{i_1,\ldots,i_k\} \in \{C_k^n\}, \quad k = \overline{2,n}, \tag{74}$$

where

$$R_{i_j}^{y_j}(x) = L(x)y_j - \int_0^{y_j} L\left(\min\left(x, \frac{u}{c_{i_j}}\right)\right)\mathrm{d}u, \quad j = \overline{1,k}.$$

Our main purpose is to obtain the steady-state customers number distribution and loss probability in the analyzed special case. First, we notice interesting property of the fuction $D_j(x)$ defined by (33) that simplify our computations. In our case, this function can be presented in the following form [19]:

$$\begin{aligned} D_j(x) &= \int_0^\infty K_j(x,u)\,\mathrm{d}u = \int_0^\infty P\{\zeta < x,\ \xi_j \geq u\}\,\mathrm{d}u \\ &= \int_0^\infty [L(x) - P\{\zeta < x,\ \xi_j < u\}]\,\mathrm{d}u \\ &= \int_0^\infty [L(x) - L(x)B_j(u|\zeta < x)]\,\mathrm{d}u \\ &= L(x)\int_0^\infty [1 - B_j(u|\zeta < x)]\,\mathrm{d}u \\ &= L(x)\cdot E(\xi_j|\zeta < x) \\ &= E(\xi_j, \zeta < x) = \int_{u=0}^x \int_{y=0}^\infty y\,\mathrm{d}F_j(u,y). \end{aligned} \tag{75}$$

If $\xi_j = c_j\zeta$, then we obtain the following formula:

$$D_j(x) = E(\xi_j, \zeta < x) = c_jE(\zeta, \zeta < x) = c_j\int_0^x u\,\mathrm{d}L(u). \tag{76}$$

If we use the generalized density function $l(x)$ of the random variable ζ [24], then we finally obtain:

$$D_j(x) = c_j\int_0^x u\,\mathrm{d}L(u) = c_j\int_0^x ul(u)\,\mathrm{d}u. \tag{77}$$

Then, the formulae (47)–(51) take the following form, as it follows from the relation (77):

$$p_1^{\{j\}} = \frac{ap_0c_j}{n}\int_0^V ul(u)\,\mathrm{d}u, \quad j = \overline{1,n}; \tag{78}$$

$$p_k^{\{i_1,\ldots,i_k\}} = \frac{a^k(n-k)!p_0}{n!}D_{i_1} * \ldots * D_{i_k}(V), \quad \{i_1,\ldots,i_k\} \in \{C_k^n\}, \quad k = \overline{2,n}, \tag{79}$$

where

$$D_{i_j}(x) = c_{i_j}\int_0^x ul(u)\,\mathrm{d}u, \quad j = \overline{1,k};$$

$$p_1 = \frac{ap_0}{n}\sum_{j=1}^n c_j\int_0^V ul(u)\,\mathrm{d}u; \tag{80}$$

$$p_k = \frac{a^k(n-k)!p_0}{n!}\sum_{\{i_1,\ldots,i_k\}\in\{C_k^n\}} D_{i_1} * \ldots * D_{i_k}(V), \quad k = \overline{2,n}; \tag{81}$$

$$p_0 = \left[1 + \frac{a}{n}\sum_{j=1}^n c_j\int_0^V u\,\mathrm{d}l(u) + \frac{1}{n!}\sum_{k=2}^n a^k(n-k)!\sum_{\{i_1,\ldots,i_k\}\in\{C_k^n\}} D_{i_1} * \ldots * D_{i_k}(V)\right]^{-1}. \tag{82}$$

Loss probability formula can be calculated using the relation (57). In this case, we also can calculate exact characteristics only if it is possible to obtain exact formulae for the convolutions appearing in (79) but it has to be also possible to obtain exact result of the integral $\int_0^x ul(u)du$. In opposite case, we can only use approximating methods.

Let us additionally assume that customer volume is exponentially distributed with parameter f and service time on j-th server is proportional to the customer volume with coefficient c_j. It means that service time on j-th server is also exponentially distributed with parameter f/c_j. After rather easy computations, we obtain in this case:

$$\int_0^x u\,\mathrm{d}l(u) = \frac{1}{f}\left[1 - (1 + fx)e^{-fx}\right]. \tag{83}$$

This formula presents the distribution function of second order Erlang distribution with parameter f multiplied by constant $\frac{1}{f}$, whereas Erlang distribution function is the Stieltjes convolution of two functions that are the distributions of independent exponentially distributed random variables with parameter f. So, we finally obtain the following formulae:

$$p_1 = \frac{ap_0}{nf}\left(1-(1+fV)e^{-fV}\right)\sum_{j=1}^{n} c_j; \tag{84}$$

$$p_k = \frac{a^k(n-k)!p_0}{n!f^k}\sum_{\{i_1,\ldots,i_k\}\in\{C_k^n\}}\prod_{j=1}^{k} c_{i_j}\left(1-e^{-fV}\sum_{l=0}^{2k-1}\frac{(fV)^l}{l!}\right), \quad k=\overline{2,n}; \tag{85}$$

$$p_0 = \left[1+\frac{a}{nf}\left(1-(1+fV)e^{-fV}\right)\sum_{j=1}^{n} c_j + \frac{1}{n!}\sum_{k=2}^{n}\left(\frac{a}{f}\right)^k (n-k)! \times \sum_{\{i_1,\ldots,i_k\}\in\{C_k^n\}}\prod_{j=1}^{k} c_{i_j}\left(1-e^{-fV}\sum_{l=0}^{2k-1}\frac{(fV)^l}{l!}\right)\right]^{-1}; \tag{86}$$

$$P_{loss} = 1-p_0\left[1-e^{-fV}+\frac{1}{n!}\sum_{k=2}^{n}\left(\frac{a}{f}\right)^{k-1}(n-k)! \times \sum_{\{i_1,\ldots,i_k\}\in\{C_k^n\}}\sum_{j=1}^{k}\prod_{l=1,\, l\neq j}^{k} c_{i_l}\left(1-e^{-fV}\sum_{l=0}^{2k-2}\frac{(fV)^l}{l!}\right)\right]. \tag{87}$$

Now, we illustrate the obtained in (84–87) theoretical results with a numerical example. Let us consider $M/\vec{M}/3/(0,V)$ queueing system in which service times are proportional to the customer volume with coefficients $c_1 = 1/2$, $c_2 = 1/4$ and $c_3 = 1/5$, respectively, and the customer volume is exponentially distributed with parameter $f = 1$. The parameter of Poisson entrance flow is equal to $a = 1$. From the classical point of view, this system is the same as the analogous one analyzed in previous subsection, but on the base of (84–87) we obtain here following different (in comparison to (70)) formulae:

$$p_1 = \frac{19}{60}p_0\left(1-e^{-V}(1+V)\right);$$

$$p_2 = \frac{11}{240}p_0\left(1-e^{-V}\left(1+V+\frac{V^2}{2}+\frac{V^3}{6}\right)\right);$$

$$p_3 = \frac{1}{240}p_0\left(1-e^{-V}\left(1+V+\frac{V^2}{2}+\frac{V^3}{6}+\frac{V^4}{24}+\frac{V^5}{120}\right)\right);$$

$$p_0 = \left[1+\frac{19}{60}\left(1-e^{-V}(1+V)\right) + \frac{11}{240}\left(1-e^{-V}\left(1+V+\frac{V^2}{2}+\frac{V^3}{6}\right)\right) + \frac{1}{240}\left(1-e^{-V}\left(1+V+\frac{V^2}{2}+\frac{V^3}{6}+\frac{V^4}{24}+\frac{V^5}{120}\right)\right)\right]^{-1};$$

$$P_{loss} = 1-p_0\left[1-e^{-V}+\frac{19}{60}\left(1-e^{-V}\left(1+V+\frac{V^2}{2}\right)\right) + \frac{11}{240}\left(1-e^{-V}\left(1+V+\frac{V^2}{2}+\frac{V^3}{6}+\frac{V^4}{24}\right)\right)\right]. \tag{88}$$

Analogously to the investigations in previous subsection, Table 5 presents the results for the steady-state customers number distribution and loss probability, obtained using (88), and Table 6 presents results for the same system obtained by simulation using DES method and with the help of programs written in Python3 language. We can see that the results of simulation (Table 6) also confirm theoretical ones (Table 5).

Table 5
Customers number distribution and loss probability for $M/\vec{M}/3/(0,V)$ system – service times are proportional to the customer volumes – theoretical results

V	p_0	p_1	p_2	p_3	P_{loss}
1	0.9220	0.0772	0.0008	0.0000	0.3936
2	0.8370	0.1574	0.0055	0.0001	0.1885
3	0.7873	0.1997	0.0127	0.0003	0.1014
4	0.7607	0.2188	0.0198	0.0007	0.0567
5	0.7467	0.2269	0.0252	0.0012	0.0321
6	0.7394	0.2301	0.0288	0.0017	0.0185
7	0.7356	0.2313	0.0310	0.0021	0.0111
8	0.7337	0.2316	0.0322	0.0025	0.0071
9	0.7327	0.2317	0.0329	0.0027	0.0051
10	0.7322	0.2317	0.0332	0.0028	0.0040

Table 6
Customers number distribution and loss probability for $M/\vec{M}/3/(0,V)$ system – service times are proportional to the customer volumes – simulation results

V	p_0^{SIM}	p_1^{SIM}	p_2^{SIM}	p_3^{SIM}	P_{loss}^{SIM}
1	0.9220	0.0772	0.0008	0.0000	0.3939
2	0.8372	0.1573	0.0054	0.0001	0.1879
3	0.7876	0.1995	0.0127	0.0003	0.1012
4	0.7609	0.2186	0.0198	0.0007	0.0567
5	0.7467	0.2269	0.0252	0.0012	0.0323
6	0.7395	0.2301	0.0287	0.0017	0.0187
7	0.7358	0.2313	0.0308	0.0021	0.0111
8	0.7334	0.2317	0.0324	0.0025	0.0070
9	0.7329	0.2317	0.0327	0.0027	0.0050
10	0.7324	0.2315	0.0333	0.0028	0.0042

In addition, we notice that for analyzed system loading parameter $\rho = a(f(1/c_1+1/c_2+1/c_3))^{-1} = \frac{1}{11} \ll 1$ (the speed of customers service is much greater than the speed of their arrival), so in this case (and also in analyzed in previous section analogous system having the same classical characteristics) loss probabilities are very small what is also connected with the small mean value of the customer volume: $E\zeta = 1$ (in comparison to V volume) and simulation results are very close to the theoretical ones. On the base of analyzed example, we can also show the possible practical applications of the obtained results. Assume that we design communication (or computer) system that has the same parameters and we only want to choose the proper value of the memory V by this way that loss probability P_{loss} satisfies the inequality $P_{loss} \leq 0.05$. In our case P_{loss} is the decreasing function of an argument V. On the base of results presented in Tables 5 and 6, we can choose the following value: $V \approx 9$. In general, if we obtain the exact formula for the loss probability: $P_{loss} = f(V)$ and it is possible to find the inverse function (it happens very seldom), we can simply chooose the memory size using formula: $V = f^{-1}(P_{loss})$. In opposite case, we can approximate this value on the base of numerical results obtained in tables.

In the next example, we analyze $M/\vec{G}/2/(0,V)$ system with Poisson entrance flow parameter $a = 4$. Assume that volumes of the arriving customers are uniformly distributed on the interval $[0,2]$ and service time for the first and second server is proportional with coefficient $c_1 = 5/2$ and $c_2 = 5$, respectively. It means that service times on both servers are also uniformly distributed on the intervals $[0,5]$ and $[0,10]$, respectively, but they depend on the customer volume. This system is the same (from the classical point of view) as analogous one analyzed in previous subsection, but on the base of (80)–(82) and (57) and with the help of *Mathematica* environment we obtain here following diiferent (in comparison to (71)) formulae:

$$\begin{aligned}
p_1 &= \frac{15}{4} p_0 \left[V^2 + (4 - V^2)H(V-2)\right]; \\
p_2 &= \frac{25}{24} p_0 \left[V(V-4)^2(V+8)H(V-4)\right. \\
&\quad \left. -2(V-2)^3(V+6)H(V-2) + V^4\right]; \\
p_0 &= \left\{1 + \frac{15}{4}\left[V^2 + (4-V^2)H(V-2)\right]\right. \\
&\quad + \frac{25}{24}\left[V(V-4)^2(V+8)H(V-4)\right. \\
&\quad \left.\left. - 2(V-2)^3(V+6)H(V-2) + V^4\right]\right\}^{-1}; \\
P_{loss} &= 1 - p_0\left\{\frac{1}{2}V - \frac{1}{2}(V-2)H(V-2)\right. \\
&\quad + \frac{15}{24}\left[(V-4)^2(V+2)H(V-4)\right. \\
&\quad \left.\left. -2(V-2)^2(V+1)H(V-2) + V^3\right]\right\}.
\end{aligned} \tag{89}$$

Theoretical and simulation results for investigated model are presented in Table 7 and Table 8, respectively.

We additionally notice that this system (and analogous system analyzed in previous subsection) is strongly overloaded (load parameter $\rho = a(1/\beta_1 + 1/\beta_2)^{-1} = \frac{20}{3} \gg 1$), so the loss probabilities are very big ($P_{loss} > 0.8$). But, in this situation, we also obtain the following convergence: $P_{loss} \to p_2$ if $V \to 4$, but simulation results are not so exact as the theoretical ones, especially for small values of V.

Table 7
Customers number distribution and loss probability for $M/\vec{G}/2/(0,V)$ system – service times and customer volumes are proportional (uniform distributions) – theoretical results

V	p_0	p_1	p_2	P_{loss}
0.5	0.4993	0.4681	0.0325	0.8362
1.0	0.1727	0.6475	0.1799	0.8058
1.5	0.0680	0.5736	0.3585	0.8056
2.0	0.0306	0.4592	0.5102	0.8163
2.5	0.0184	0.2753	0.7063	0.8225
3.0	0.0123	0.1838	0.8040	0.8423
3.5	0.0095	0.1422	0.8484	0.8565
4.0	0.0086	0.1293	0.8621	0.8621

Table 8
Customers number distribution and loss probability for $M/\vec{G}/2/(0,V)$ system – service times and customer volumes are proportional (uniform distributions) – simulation results

V	p_0^{SIM}	p_1^{SIM}	p_2^{SIM}	P_{loss}^{SIM}
0.5	0.5001	0.4674	0.0326	0.8363
1.0	0.1727	0.6478	0.1795	0.8060
1.5	0.0680	0.5736	0.3584	0.8056
2.0	0.0305	0.4578	0.5116	0.8161
2.5	0.0183	0.2748	0.7069	0.8225
3.0	0.0124	0.1836	0.8040	0.8423
3.5	0.0095	0.1422	0.8483	0.8566
4.0	0.0086	0.1293	0.8621	0.8619

Finally, if we compare the numerical results obtained in the above subsections, we can easily notice that two pairs of models under analysing do not differ from the classical point of view. It means that they have the same classical characteristics such like service time distributions or parameters of Poisson entrance flow. But formula (70) differs from (88) and formula (71) differs from (89). This fact shows that the character of dependency between customer volume and his service time has an influence on steady-state customers number distribution and loss probability. Our calculations appearing in Tables 1–8 confirm this fact. For example, loss probabilities have bigger values in the case when service time is independent on customer volume.

We can also notice that $P_{loss} \to p_n$ when $V \to \infty$ which is obvious, because the big values V have a small influence on the value of loss probability, that begins to depend more and more on the number of servers.

5. Conclusions

In the present paper, we investigate the queueing system of $M/G/n/(0,V)$-type with heterogeneous servers, nonhomogeneous customers and limited memory space. In the beginning, after exact analysis, we obtain the formulae for steady-state customers number distribution and loss probability. Later on, we investigate two interesting and practical special cases of the analyzed model: when service time on every server and customer volume are independent and when service time is proportional to the customer volume for these servers. We illustrate these investigations with some numerical examples. Both the theoretical and simulation results show that the steady-state customers number distribution and loss probability depend on the form of the joint distribution of the customer volume and his service time, and this dependency has a substantial influence on the steady-state characteristics. Obtained results can be exemplary applied in the process of computer system designing to calculate the size of needed memory volume.

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