10.2478/v10170-011-0001-7

Archives of Control Sciences
Volume 21(LVII), 2011
No. 4, *pages 349–361*

Regular design equations for the continuous reduced-order Kalman filter

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Reduced-order Kalman filters yield an optimal state estimate for linear dynamical systems, where parts of the output are not corrupted by noise. The design of such filters can either be carried out in the time domain or in the frequency domain. Different from the full-order case where all measurements are noisy, the design equations of the reduced-order filter are not regular. This is due to the rank deficient measurement covariance matrix and it can cause problems when using standard software for the solution of the Riccati equations in the time domain. In the frequency domain the spectral factorization of the non-regular polynomial matrix equation does not cause problems. However, the known proof of optimality of the factorization result also requires a regular measurement covariance matrix. This paper presents regular (reduced-order) design equations for reduced-order Kalman filters in the time and in the frequency domains for linear continuous-time systems. They allow to use standard software for the design of the filter, to formulate the conditions for the stability of the filter and they also prove that the existing frequency domain solutions obtained by spectral factorization of a non-regular polynomial matrix equation are indeed optimal.

Key words: optimal estimation, polynomials, multivariable systems, continuous-time systems

1. Introduction

If the system is completely observable the dynamics of a state observer can be assigned arbitrarily. In the absence of disturbances the observer generates a state estimate $\hat{x}$ that converges towards the real state x of the system. In the presence of stochastic disturbances, however, persistent observation errors occur. Then, a state estimate is of interest such that the observation error $\hat{x} - x$ has the smallest mean square. Given Gaussian white noise with zero mean, such an estimate is generated by a stationary Kalman filter ([1], [10]) whose order coincides with the order n of the plant.

If κ of the measurements are not corrupted by noise, the order of the optimal filter is reduced to $n - \kappa$. The optimal estimation problem in the presence of partly noise-free measurements is one of the well researched fields in automatic control. Since the

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Received 18.11.2011.

original work [2] a considerable amount of contributions has been published on the subject (see, *e.g.*, the books [14], [4], [12] and [8], or the references in [11] and [3]). The time-domain design of the reduced-order filter amounts to solving an algebraic Riccati equation (ARE).

The equivalent frequency domain version of the reduced-order Kalman filter is parameterized by a polynomial matrix $\tilde{\bar{D}}(s)$, which can be obtained by spectral factorization of a polynomial matrix equation. This polynomial matrix equation can be derived from a version of the ARE introduced, *e.g.*, in [2] and [4]. This ARE is formulated for a $n \times n$ covariance matrix $\bar{P}$ which, however, is singular in the presence of noise-free measurements. Not formulated so far were the requirements on the system so that the resulting optimal reduced-order filter is stable. There have been papers presenting regular reduced-order Riccati equations, which yield a regular $(n-\kappa) \times (n-\kappa)$ covariance matrix $\bar{P}_r$, but they cannot be used to develop an equivalent frequency domain formulation of the filtering problem.

Standard software cannot be used to design the reduced-order Kalman filter, because the basic requirement, namely a measurement covariance matrix which is positive definite, is not fulfilled in the presence of undisturbed measurements. To obtain a well-defined order of the reduced-order filter it is assumed here that the random signals which disturb the artificial output, consisting of the noisy measurements and the time derivatives of the undisturbed outputs, have a regular covariance. This is a standard assumption in nearly all investigations on reduced-order Kalman filters (see, *e.g.*, [2], [12], [6], [7]).

After a formulation of the underlying problem in the time domain in Section 2 the existing solution for the optimal filter is presented. By a reformulation of the Riccati equation for the artificial output, one obtains a regular measurement covariance. In the continuous-time case standard software still does not work because the Hamiltonian of this ARE has eigenvalues at $s = 0$. After an adequate transformation of the state equations of the system this Riccati equation can be subdivided into a regular part and a vanishing part. The regular part represents a full-order filter for a reduced-order system of the order $n - \kappa$, which is readily solvable by standard software. This full-order filtering problem for a reduced-order system also allows to derive conditions for the optimal filter to be stable, and it is shown how these conditions translate into conditions on the original system.

The known polynomial matrix equation for the design of the reduced-order Kalman filter in the frequency domain is based on the left MFD of the full-order system whereas the polynomial matrix $\tilde{\bar{D}}(s)$, resulting from the spectral factorization of this polynomial matrix characterizes a system of reduced order. This is a consequence of the rank deficient measurement covariance matrix multiplying the denominator matrix of the system. Unfortunately, proofs for the optimality of the spectral factor are only known in the case, where the measurement covariance is not singular. In [8] it has been observed that, on the one hand, optimality of the result can only be checked by computing the corresponding time domain results and, on the other hand, that all examples investigated so far have shown that the resulting $\tilde{\bar{D}}(s)$ is indeed optimal.

In Section 3 it is shown that the polynomial matrix $\tilde{\tilde{D}}(s)$ resulting from the non-regular polynomial matrix equation can also be obtained from a "regular" polynomial matrix equation. This regular polynomial matrix equation is derived from the reduced regular ARE in the time domain. As an additional result, the conditions on the MFD of the system are presented, which guarantee the stability of the reduced-order filter in a frequency domain design.

Concluding remarks are presented in Section 4

2. The filter design in the time domain

We consider linear time-invariant systems of the order n, with p inputs u, q stochastic inputs w and m measured outputs y, where the first $m-\kappa$ outputs y_1 are corrupted by noise and the remaining κ outputs y_2 are free of noise, described by

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t) \tag{1}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x(t) + \begin{bmatrix} v_1(t) \\ 0 \end{bmatrix} \tag{2}$$

where the abbreviation

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = C \tag{3}$$

will be used in the sequel. It is assumed that the system is controllable from the input w and that it is observable.

The stochastic inputs $w \in \mathbb{R}^q$ and $v_1 \in \mathbb{R}^{m-\kappa}$ are independent, zero-mean, stationary Gaussian white noises with

$$\mathrm{E}\{w(t)w^T(\tau)\} = \bar{Q}\delta(t-\tau) \tag{4}$$

$$\mathrm{E}\{v_1(t)v_1^T(\tau)\} = \bar{R}_1\delta(t-\tau) \tag{5}$$

where $\mathrm{E}\{\cdot\}$ denotes the mathematical expectation and $\delta(t)$ is the Dirac delta function.

The covariance matrices $\bar{Q}$ and $\bar{R}_1$ are real and symmetric, where $\bar{Q}$ is positive-semidefinite and $\bar{R}_1$ is positive-definite. The initial state $x(0) = x_0$ is not correlated with the disturbances, *i.e.*, $\mathrm{E}\{x_0 w^T(t)\} = 0$ and $\mathrm{E}\{x_0 v_1^T(t)\} = 0$ for all $t \geqslant 0$.

It is assumed that the covariance matrix

$$\Phi = C_2 G \bar{Q} G^T C_2^T \tag{6}$$

is positive definite. It characterizes the influence of the input noise on the time derivative of the undisturbed measurement y_2.

Consider the $n-\kappa$ linear combinations

$$\zeta(t) = Tx(t) \tag{7}$$

and the κ ideal measurements y_2, which can be used to represent the state x of the system as

$$x(t) = \begin{bmatrix} C_2 \\ T \end{bmatrix}^{-1} \begin{bmatrix} y_2(t) \\ \zeta(t) \end{bmatrix} = \Psi_2 y_2(t) + \Theta\zeta(t) \tag{8}$$

Then the reduced-order Kalman filter for such systems is described by

$$\dot{\hat{\zeta}}(t) = T(A-L_1C_1)\Theta\hat{\zeta}(t) + [TL_1 \quad T(A-L_1C_1)\Psi_2]\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + TBu(t) \tag{9}$$

$$\hat{x}(t) = \Theta\hat{\zeta}(t) + \Psi_2 y_2(t) \tag{10}$$

(see [4], [8]). The optimal estimate $\hat{\zeta}(t)$ results if the matrices L_1 and Ψ_2 are chosen such that

$$L_1 = \bar{P}C_1^T\bar{R}_1^{-1} \tag{11}$$

and

$$\Psi_2 = \left(\bar{P}A^TC_2^T + G\bar{Q}G^TC_2^T\right)\Phi^{-1} \tag{12}$$

with Φ as in (6) and $\bar{P} = \bar{P}(\infty)$ defined by

$$\bar{P}(t) = \mathrm{E}\{(x(t)-\hat{x}(t))(x(t)-\hat{x}(t))^T\} \tag{13}$$

The stationary covariance $\bar{P}$ satisfies the ARE

$$A\bar{P} + \bar{P}A^T - \begin{bmatrix} L_1 & \Psi_2 \end{bmatrix}\begin{bmatrix} \bar{R}_1 & 0 \\ 0 & \Phi \end{bmatrix}\begin{bmatrix} L_1^T \\ \Psi_2^T \end{bmatrix} + G\bar{Q}G^T = 0 \tag{14}$$

([8]) and this version of the ARE can be used as a starting point for deriving the equivalent frequency-domain solution of the filter (see Section 3). This ARE, however, is not in a standard form to be solved for $\bar{P}$.

Inserting the optimal solutions (14) and (16) in (18) one obtains

$$\widetilde{A}\bar{P} + \bar{P}\widetilde{A}^T - \bar{P}\widetilde{C}^T\widetilde{R}^{-1}\widetilde{C}\bar{P} + G\widetilde{Q}G^T = 0 \tag{15}$$

with

$$\widetilde{A} = A - G\bar{Q}G^TC_2^T\Phi^{-1}C_2A \tag{16}$$

$$\widetilde{C} = \begin{bmatrix} C_1 \\ C_2A \end{bmatrix} \tag{17}$$

$$\widetilde{R} = \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & \Phi \end{bmatrix} \tag{18}$$

and

$$\widetilde{Q} = \bar{Q} - \bar{Q}G^T C_2^T \Phi^{-1} C_2 G \bar{Q} \tag{19}$$

The ARE (19) is in the standard form with a regular $\widetilde{R} > 0$. Standard software as, *e.g.*, the function *lqe* in MATLAB®, however, does not yield the solution $\bar{P}$, because the Hamiltonian related to the ARE (19) has eigenvalues at $s = 0$. This is due to the fact that $\operatorname{rank} \bar{P} = n - \kappa$.

By a regular state transformation $z(t) = \bar{T}x(t)$ with

$$\bar{T} = \begin{bmatrix} * \\ C \end{bmatrix} \tag{20}$$

the state equations (1)–(5) of the system can always be transformed into

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}u(t) + \bar{G}w(t) \tag{21}$$

$$y(t) = \bar{C}z(t) + \begin{bmatrix} v_1(t) \\ 0 \end{bmatrix} \tag{22}$$

with

$$\bar{A} = \bar{T}A\bar{T}^{-1},\ \bar{B} = \bar{T}B,\ \bar{G} = \bar{T}G,\ \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix} = C\bar{T}^{-1} = \begin{bmatrix} 0 & I_{m-\kappa} & 0 \\ 0 & 0 & I_\kappa \end{bmatrix} \tag{23}$$

or in components

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{22} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} w(t) \tag{24}$$

$$y_1(t) = \begin{bmatrix} \check{C}_1 & 0 \end{bmatrix} z(t) + v_1(t) \tag{25}$$

$$y_2(t) = \begin{bmatrix} 0 & I_\kappa \end{bmatrix} z(t) \tag{26}$$

with $z_1 \in \mathbb{R}^{n-\kappa}$, $0 < \kappa \leqslant m$, $z_2 \in \mathbb{R}^\kappa$ and $\check{C}_1 = [0 \quad I_{m-\kappa}]$.

If the transformed matrices (23) are inserted in (19)–(19) the solution $\bar{P}_z = \bar{T}\bar{P}\bar{T}^T$ of the ARE (19) has the form

$$\bar{P}_z = \begin{bmatrix} \bar{P}_r & 0 \\ 0 & 0_\kappa \end{bmatrix} \tag{27}$$

and the ARE (19) then consists of a regular (upper left) part

$$A_r\bar{P}_r + \bar{P}_r A_r^T - \bar{P}_r C_r^T \widetilde{R}^{-1} C_r \bar{P}_r + G_r \widetilde{Q} G_r^T = 0 \tag{28}$$

while the rest is vanishing. The matrices in (32) are defined by

$$A_r = A_{11} - G_1 \bar{Q} G_2^T \Phi^{-1} A_{21} \tag{29}$$

$$G_r = G_1 \tag{30}$$

and

$$C_r = \begin{bmatrix} \breve{C}_1 \\ A_{21} \end{bmatrix} \tag{31}$$

so that the reduced-order Kalman filter can be regarded as a regular full-order filter for the reduced system (A_r, G_r, C_r). The feedback matrix L_r is defined by

$$L_r = \bar{P}_r C_r^T \widetilde{R}^{-1} = \bar{P}_r \begin{bmatrix} \breve{C}_1^T \bar{R}_1^{-1} & A_{21}^T \Phi^{-1} \end{bmatrix} \tag{32}$$

The ARE (32) has two advantages. First, it can be used to obtain $\bar{P}_r$ and consequently also $\bar{P}$ by standard software. Second, it defines the conditions, which guarantee a stable filter. It is known that the full-order Kalman filter for the system (A_r, G_r, C_r) is stable if the pair $(A_r, G_r \widetilde{Q}_0)$ has no uncontrollable eigenvalues on the imaginary axis, where

$$\widetilde{Q} = \widetilde{Q}_0 \widetilde{Q}_0^T \tag{33}$$

([5]). Introducing

$$\bar{Q} = \bar{Q}_0 \bar{Q}_0^T \tag{34}$$

and

$$\hat{Q} = I - \bar{Q}_0^T G_2^T \Phi^{-1} G_2 \bar{Q}_0 \tag{35}$$

it is easy to show that

$$\widetilde{Q}_0 = \bar{Q}_0 \hat{Q} \tag{36}$$

when taking

$$C_2 G = \bar{C}_2 \bar{G} = G_2 \tag{37}$$

into account. Given the above condition for a stable filter in terms of A_r and G_r, it is of interest to know the corresponding condition for the non-reduced system $(\bar{A}, \bar{G}, \bar{C})$. The answer is contained in the following lemma.

Lemma 1 *If the system*

$$\dot{z}(t) = \bar{A} z(t) + \bar{G} \bar{Q}_0 w(t) \tag{38}$$

$$y_2(t) = \begin{bmatrix} 0 & I_\kappa \end{bmatrix} z(t) \tag{39}$$

does not have zeros, which are located on the imaginary axis, then the pair $(A_r, G_r \widetilde{Q}_0)$ *has no uncontrollable eigenvalues on the imaginary axis and vice versa.*

Proof: If $s = s_i$ is a non-controllable eigenvalue of the pair $(A_r, G_r\widetilde{Q}_0)$ then

$$\text{rank}\left[\begin{array}{c} s_i I - A_r \vdots \;\; G_r\widetilde{Q}_0 \end{array}\right] < n - \kappa \tag{40}$$

(see, *e.g.*, [9]).

Now define the system matrix

$$P(s) = \begin{bmatrix} sI_{n-\kappa} - A_{11} & -A_{12} & G_1\bar{Q}_0 \\ -A_{21} & sI_\kappa - A_{22} & G_2\bar{Q}_0 \\ 0 & -I_\kappa & 0 \end{bmatrix} \tag{41}$$

which characterizes the zeros of the system (38)–(39) (see [13]). If the system (38)–(39) has a zero at $s = s_i$, then rank $P(s_i) < n + \kappa$.

Using the unimodular matrix

$$U_L = \begin{bmatrix} I_{n-\kappa} & -G_1\bar{Q}G_2^T\Phi^{-1} & 0 \\ 0 & I_\kappa & 0 \\ 0 & 0 & I_\kappa \end{bmatrix} \tag{42}$$

and the unimodular matrix

$$U_R = \begin{bmatrix} I_{n-\kappa} & 0 & 0 \\ 0 & I_\kappa & 0 \\ \bar{Q}_0^T G_2^T \Phi^{-1} A_{21} & 0 & I_q \end{bmatrix} \tag{43}$$

one obtains

$$U_L P(s_i) U_R = \begin{bmatrix} s_i I - A_r & * & G_r\widetilde{Q}_0 \\ 0 & * & G_2\bar{Q}_0 \\ 0 & -I_\kappa & 0 \end{bmatrix} \tag{44}$$

Since it has been assumed that rank $G_2\bar{Q}_0 = \kappa$ (see (6)) the result (44) shows that the system (38)–(39) has a zero at $s = s_i$ if and only if $s = s_i$ is an uncontrollable eigenvalue in the pair $(A_r, G_r\widetilde{Q}_0)$ and *vice versa*. This is, of course, not only true for the transformed system (38)–(39) but also for the original system $(A, G\bar{Q}_0, C_2)$.

3. The filter design in the frequency domain

The system (1)–(5) or (21)–(22) is described in the frequency domain by

$$y(s) = F(s)w(s) + \begin{bmatrix} v_1(s) \\ 0 \end{bmatrix} \tag{45}$$

with

$$F(s) = C(sI - A)^{-1}G = \bar{C}(sI - \bar{A})^{-1}\bar{G} \tag{46}$$

Given the left coprime MFD

$$F(s) = \bar{D}^{-1}(s)\bar{N}_w(s) \tag{47}$$

the reduced-order Kalman filter is parameterized by the polynomial matrix $\widetilde{\bar{D}}(s)$ resulting by spectral factorization of the right hand side of

$$\widetilde{\bar{D}}(s)\widetilde{R}\widetilde{\bar{D}}^T(-s) = \bar{D}(s)\begin{bmatrix} \bar{R}_1 & 0 \\ 0 & 0 \end{bmatrix}\bar{D}^T(-s) + \bar{N}_w(s)\bar{Q}\bar{N}_w^T(-s) \tag{48}$$

where

$$\Gamma_r\left[\widetilde{\bar{D}}(s)\right] = \Gamma_r\left[\bar{D}_\kappa(s)\right] \tag{49}$$

with the row-reduced polynomial matrix

$$\bar{D}_\kappa(s) = \Pi\left\{\bar{D}(s)\begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & s^{-1}I_\kappa \end{bmatrix}\right\} \tag{50}$$

(see [8], [7]). Here, $\Gamma_r[\cdot]$ denotes the highest row-degree-coefficient matrix and $\Pi[\cdot]$ taking the polynomial part.

The polynomial matrix $\widetilde{\bar{D}}(s)$ is related with the time domain parameters by

$$\bar{D}^{-1}(s)\widetilde{\bar{D}}(s) = \bar{C}(sI - \bar{A})^{-1}[L_1 \quad \Psi_2] + \begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & 0_\kappa \end{bmatrix} \tag{51}$$

where L_1 and Ψ_2 are defined by (14) and (16) and they obtain the special forms

$$L_1 = \begin{bmatrix} \bar{P}_r\check{C}_1^T \\ 0 \end{bmatrix}\bar{R}_1^{-1} \tag{52}$$

and

$$\Psi_2 = \begin{bmatrix} \left(\bar{P}_r A_{21}^T + G_1\bar{Q}G_2^T\right)\Phi^{-1} \\ I_\kappa \end{bmatrix} \tag{53}$$

when using the system description (21)–(11).

In [8] the solution (48)–(50) is presented without rigorous proof, because the polynomial matrix equation (48) contains a singular measurement covariance matrix on the right hand side, and the known proofs of optimality of a $\widetilde{\bar{D}}(s)$ obtained by spectral factorization are for full-order filters with regular measurement covariance matrices.

The polynomial matrix equation (48) was derived on the basis of the ARE (18). As shown in Section 2, the reduced-order Kalman filter can also be designed on the basis of the regularized ARE (32) with a regular measurement covariance matrix $\widetilde{R}$, *i.e.*, one can design the reduced-order Kalman filter as a regular full-order filter for the reduced system (A_r, C_r, G_r).

Introducing the left coprime MFD of

$$F_r(s) = C_r(sI - A_r)^{-1}G_r \tag{54}$$

namely

$$F_r(s) = \bar{D}_r^{-1}(s)\bar{N}_{wr}(s) \tag{55}$$

and the polynomial matrix $\widetilde{\bar{D}}_r(s)$ parameterizing the full-order Kalman filter related to the parameters $(A_r, G_r, C_r, \bar{P}_r)$ according to

$$\bar{D}_r^{-1}(s)\widetilde{\bar{D}}_r(s) = C_r(sI - A_r)^{-1}L_r + I_m \tag{56}$$

(see [8], [7]), the Riccati equation (32) can be transformed into the polynomial matrix equation

$$\widetilde{\bar{D}}_r(s)\widetilde{R}\widetilde{\bar{D}}_r^T(-s) = \bar{D}_r(s)\widetilde{R}\bar{D}_r^T(-s) + \bar{N}_{wr}(s)\widetilde{Q}\bar{N}_{wr}^T(-s) \tag{57}$$

by similar steps as in the derivation of (48) from (18) in [8]. This is a regular polynomial matrix equation with $\widetilde{R} > 0$ and consequently the polynomial matrix $\widetilde{\bar{D}}_r(s)$ obtained by spectral factorization of the right hand side of (57) with

$$\Gamma_r\left[\widetilde{\bar{D}}_r(s)\right] = \Gamma_r\left[\bar{D}_r(s)\right] \tag{58}$$

(see [8], [7]) parameterizes the optimal full-order Kalman filter for the reduced-order system (55) in the frequency domain.

If this $\widetilde{\bar{D}}_r(s)$ is identical with $\widetilde{\bar{D}}(s)$ obtained from the spectral factorization of (48), it follows that the solution procedure presented in [8] yields indeed the optimal results.

Given the transformed system description (21),(22) and the MFD (47), *i.e.*, a denominator matrix $D(s)$ such that $\bar{D}_\kappa(s)$ as defined in (50) is row reduced. Then define the MFD

$$\bar{C}(sI - \bar{A})^{-1} = \bar{D}^{-1}(s)\bar{N}_z(s) \tag{59}$$

with $\bar{D}(s)$ as in (47) and $\bar{N}_z(s)$ partitioned according to

$$\bar{N}_z(s) = [\bar{N}_{z1}(s) \quad \bar{N}_{z2}(s)] \tag{60}$$

where $\bar{N}_{z1}(s)$ has $n - \kappa$ columns and $\bar{N}_{z2}(s)$ has κ columns.

Theorem 1 *The polynomial matrix $\tilde{\bar{D}}_r(s)$ resulting from (57) is identical with $\tilde{\bar{D}}(s)$ resulting from (48) if the polynomial matrices in the MFD (55) are chosen as*

$$\bar{N}_{wr}(s) = \bar{N}_{z1}(s)G_1 \tag{61}$$

and

$$\bar{D}_r(s) = [\bar{N}_{z1}(s)\ \ \bar{N}_{z2}(s)] \begin{bmatrix} 0_{n-\kappa,m-\kappa} & G_1\bar{Q}G_2^T\Phi^{-1} \\ 0_{\kappa,m-\kappa} & I_\kappa \end{bmatrix} + \bar{D}(s) \begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & 0_\kappa \end{bmatrix} \tag{62}$$

The polynomial matrix $\tilde{\bar{D}}(s) = \tilde{\bar{D}}_r(s)$ parameterizes a stable filter if the pair

$$\left(\bar{D}(s) \begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & 0_\kappa \end{bmatrix},\ \bar{N}_w(s)\bar{Q}_0 \right) \tag{63}$$

has no greatest common left devisor with zeros on the imaginary axis.

Proof: Comparing (54) and (55) and observing (61) and (30) one can conclude that for arbitrary G_1

$$C_r(sI - A_r)^{-1} = \bar{D}_r^{-1}(s)\bar{N}_{z1}(s) \tag{64}$$

Writing (37) in the modified form $\bar{D}(s)\bar{C} = \bar{N}_z(s)(sI - \bar{A})$ and using (6) – (11) one obtains

$$\bar{D}(s) \begin{bmatrix} \breve{C}_1 \\ 0 \end{bmatrix} = \bar{N}_{z1}(s)(sI - A_{11}) - \bar{N}_{z2}(s)A_{21} \tag{65}$$

This allows to show that $\bar{N}_{z1}(s)(sI - A_r) = \bar{D}_r(s)C_r$, which then proves that the pair (61) and (62) constitutes a left MFD of (54). Inserting (61) and (62) in (57) it is straightforward to show that the right hand sides of the polynomial equations (48) and (57) coincide, so that $\tilde{\bar{D}}(s)\tilde{R}\tilde{\bar{D}}^T(-s) = \tilde{\bar{D}}_r(s)\tilde{R}\tilde{\bar{D}}_r^T(-s)$. Since $\tilde{R}$ is positive definite, this shows that $\tilde{\bar{D}}(s) = \tilde{\bar{D}}_r(s)$.

The equality $\tilde{\bar{D}}(s) = \tilde{\bar{D}}_r(s)$ can also be installed by comparing

$$\tilde{\bar{D}}(s) = \bar{N}_z(s) \begin{bmatrix} L_1 & \Psi_2 \end{bmatrix} + \bar{D}(s) \begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & 0_\kappa \end{bmatrix} \tag{66}$$

which is another form of (51) and

$$\tilde{\bar{D}}_r(s) = \bar{N}_{z1}(s)L_r + \bar{D}_r(s) \tag{67}$$

which results from (56) and then using (32), (52), (53) and (62). This proves the first part of the theorem.

Since (57) represents a regular full-order filter problem for the reduced system (A_r, G_r, C_r), the filter parameterized by $\widetilde{\bar{D}}_r(s)$ is optimal and stable if the pair

$$\left(\bar{D}_r(s)\ ,\ \bar{N}_{wr}(s)\widetilde{Q}_0\right) \tag{68}$$

has no common greatest left devisor $U_L(s)$ with zeros on the imaginary axis ([5]).

Two polynomial matrices are relatively left coprime if they meet the Bezout identity. If they contain a non-unimodular greatest common left devisor $U_L(s)$, the identity matrix is replaced by $U_L(s)$ ([9]).

If the pair (68) contains a non-unimodular greatest common left devisor $U_L(s)$ there exist solutions $\bar{Y}_{0r}(s)$ and $\bar{X}_{0r}(s) = \begin{bmatrix} \bar{X}_{0r1}(s) \\ \bar{X}_{0r2}(s) \end{bmatrix}$ of the Diophantine equation

$$\bar{N}_{z1}(s)G_1\widetilde{Q}_0\bar{Y}_{0r}(s) + \bar{D}_r(s)\begin{bmatrix} \bar{X}_{0r1}(s) \\ \bar{X}_{0r2}(s) \end{bmatrix} = U_L(s) \tag{69}$$

(see, *e.g.*, [8]).

If, on the other hand, the pair (63) contains a non-unimodular greatest common left devisor $U_L(s)$ there exist solutions $\bar{Y}_0(s)$ and $\bar{X}_0(s) = \begin{bmatrix} \bar{X}_{01}(s) \\ \bar{X}_{02}(s) \end{bmatrix}$ of the Diophantine equation

$$\left[\bar{N}_{z1}(s)G_1 + \bar{N}_{z2}(s)G_2\right]\bar{Q}_0\bar{Y}_0(s) + \bar{D}(s)\begin{bmatrix} I & 0 \\ 0 & 0_\kappa \end{bmatrix}\begin{bmatrix} \bar{X}_{01}(s) \\ \bar{X}_{02}(s) \end{bmatrix} = U_L(s) \tag{70}$$

where the fact has been exploited that $\bar{N}_w(s) = \bar{N}_z(s)\bar{G}$ (compare (37) with (46) and (47)). Given the solutions $\bar{Y}_{0r}(s)$ and $\bar{X}_{0r}(s)$ of (69) the polynomial matrices

$$\bar{X}_{01}(s) = \bar{X}_{0r1}(s) \tag{71}$$

$$\bar{X}_{02}(s) = 0 \tag{72}$$

and

$$\bar{Y}_0(s) = \hat{Q}\bar{Y}_{0r}(s) + \bar{Q}_0^T G_2^T \Phi^{-1}\bar{X}_{0r2}(s) \tag{73}$$

solve the equation (70).

Given the solutions $\bar{Y}_0(s)$ and $\bar{X}_0(s)$ of (70) the polynomial matrices

$$\bar{X}_{01r}(s) = \bar{X}_{01}(s) \tag{74}$$

$$\bar{X}_{0r2}(s) = G_2\bar{Q}_0\bar{Y}_0(s) \tag{75}$$

and

$$\bar{Y}_{0r}(s) = \hat{Q}\bar{Y}_0(s) \tag{76}$$

solve the equation (69). This shows that if the pair (68) does not contain a greatest common left devisor with zeros on the imaginary axis, then also the pair (63) does not contain such a greatest common left devisor and *vice versa*. This proves the second part of the theorem.

4. Conclusions

Some open problems in the design of reduced-order Kalman filters for linear continuous-time systems have been solved. Due to the singular measurement covariance matrix standard software cannot be used to solve the ARE of the reduced-order filter. By defining an artificial output of the system, a form of the ARE can be obtained, which exhibits a regular measurement covariance matrix. However, also this form is not solvable by the standard routines, as the corresponding Hamiltonian has eigenvalues at $s = 0$. By applying an appropriate state transformation to the original system, a modified form of the ARE results, which can be subdivided into a regular part and a vanishing part. The regular part is readily solvable for the matrix $\bar{P}$, parameterizing the filter in the time domain, and this regular part also characterizes the conditions, which guarantee a stable filter. These conditions for the parameters of the reduced-order system have been translated into conditions for the original full-order system.

The polynomial matrix equation defining the parameterizing polynomial matrix of the reduced-order filter in the frequency domain contains a singular measurement covariance matrix. This does not cause problems when applying spectral factorization to obtain the parameterizing polynomial matrix of the reduced-order filter. However, neither a proof of optimality nor the conditions for the stability of the filter were known so far. Based on the reduced-order model of the system in the time domain, a regular full-order filter design also becomes possible in the frequency domain. This allows to prove the optimality of the results obtained by the known non-regular factorization and it also allows to formulate the conditions on the MFD of the original full-order system, which guarantee a stable filter.

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