10.1515/acsc-2017-0001

Archives of Control Sciences
Volume 27(LXIII), 2017
No. 1*, pages 5–27*

Rational taxation in an open access fishery model

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We consider a model of fishery management, where n agents exploit a single population with strictly concave continuously differentiable growth function of Verhulst type. If the agent actions are coordinated and directed towards the maximization of the discounted cooperative revenue, then the biomass stabilizes at the level, defined by the well known "golden rule". We show that for independent myopic harvesting agents such optimal (or ε-optimal) cooperative behavior can be stimulated by the proportional tax, depending on the resource stock, and equal to the marginal value function of the cooperative problem. To implement this taxation scheme we prove that the mentioned value function is strictly concave and continuously differentiable, although the instantaneous individual revenues may be neither concave nor differentiable.

Key words: marginal value function, stimulating taxes, myopic agents, optimal control.

1. Introduction

An unregulated open access to marine resources, where many individual users are involved in the fishery, may easily lead to the over-exploitation or even extinction of fish populations. Moreover, it results in zero rent. These negative consequences of the unregulated open access (the "tragedy of commons": [13]) were widely discussed in the literature: see [11, 6, 8, 2]. Maybe the most evident reason for the occurrence of these phenomena is the myopic behavior of competing harvesting agents, who are interested in the maximization of instantaneous profit flows, and not in the conservation of the population in the long run. In the present paper we consider the problem of rational regulation of an open access fishery, using taxes as the only economical instrument. Other known instruments include fishing quotas of different nature, total allowable catch, limited entry, sole ownership, community rights, various economic restrictions, etc: see, e.g, [8, 2].

We should also mention that there is a natural and popular approach to modeling resource exploitation via the dynamic games. This approach is not touched in the present paper, we only refer to [19] for a survey.

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The research is supported by Southern Federal University, project 213.01-07-2014/07.
Received 24.10.2016.

Assume for a moment that n agents coordinate their efforts to maximize the aggregated long-run discounted profit. The related aggregated agent, which can be considered as a sole owner of marine fishery resources, conserves the resource under optimal strategy, unless the discounting rate is very large. How such an acceptable cooperative behavior can be realized in practice?

We consider the following scheme. Suppose that some regulator (e.g., the coastal states), being aware of the revenue function and maximal productivity of each agent, declares the amount of proportional tax on catch. Roughly speaking, it turns out that if this tax is equal to the marginal indirect utility (marginal value function) of the cooperative optimization problem, then the myopic profit maximizing agents will follow an optimal cooperative strategy, maximizing the aggregated long-run discounted profit. The idea of using such taxes in harvesting management was often expressed in the bioeconomic literature: see [7], [20], [12, Chapter 10], [15, Chapter 7]. Our goal is to study this idea more closely from the mathematical point of view.

The first theoretical question we encounter, trying to implement the mentioned taxation scheme, concerns the differentiability of the value function v of the cooperative problem. Assuming that the population growth function is strictly concave and continuously differentiable, in Sections 2 and 3 we prove v inherits these properties, although the instantaneous revenue functions may be non-concave.

The differentiability of v is proved by the tools from optimal control and convex analysis. Our approach relies on the characterization of v as the unique solution of the related Hamilton-Jacobi-Bellman equation. We neither use the general results like [22], nor the related technique. At the same time, our results are not covered by [22]. Simultaneously we construct optimal strategies and prove that optimal trajectories are attracted to the biomass level $\widehat{x}$, defined by the well known "golden rule". This level depends on the discounting rate, which is at regulator's disposal.

If the agent revenue function are non-concave, then an optimal solution of the infinite horizon cooperative problem may exist only in the class of relaxed (or randomized) harvesting strategies. Such strategies can hardly be realized in practice, and certainly cannot be stimulated by taxes. Nevertheless, in Section 4 we show that piecewise constant strategies (known as the "pulse fishing") of myopic agents, stimulated by the proportional tax $v'\alpha$ on the fishing intensity α, are ε-optimal for the cooperative problem. Moreover, the related trajectory is retained in any desired neighbourhood of $\widehat{x}$ for large values of time. Finally, we introduce the notion of the critical tax $v'(\widehat{x})$ and prove that it can only increase, when the agent community widens.

2. Cooperative harvesting problem: the case of concave revenues

Let a population biomass X satisfy the differential equation

$$X_t = x + \int_0^t b(X_s)\, ds - \sum_{i=1}^n \int_0^t \alpha_s^i\, ds, \tag{1}$$

where b is the growth rate of the population, and α^i is the harvesting rate of i-th agent. We assume that b is a *differentiable strictly concave* function defined on an open neighbourhood of $[0,1]$, and

$$b(x) > 0, \quad x \in (0,1), \quad b(0) = b(1) = 0.$$

The widely used Verhulst growth function $b(x) = x(1-x)$ is a typical example. Agent harvesting strategies α^i are (Borel) measurable functions with values in the intervals $[0,\overline{\alpha}^i]$, $\overline{\alpha}^i > 0$. A harvesting strategy $\alpha = (\alpha^1,\dots,\alpha^n)$ is called *admissible* if the solution $X^{x,\alpha}$ of (1) stays in $[0,1]$ forever: $X_t^{x,\alpha} \in [0,1]$, $t \geqslant 0$. Note that for given α the solution $X^{x,\alpha}$ is unique, since b, being concave, is Lipschitz continuous. The set of admissible strategies, corresponding to an initial condition x, is denoted by $\mathscr{A}_n(x)$.

Consider the cooperative objective functional

$$J_n(x,\alpha) = \sum_{i=1}^n \int_0^\infty e^{-\beta t} f_i(\alpha_t^i)\, dt, \quad \beta > 0$$

of the agent community. *We always assume* that the instantaneous revenue function $f_i : [0,\overline{\alpha}^i] \mapsto \mathbb{R}_+$ of i-th agent is at least *continuous*, and $f_i(0) = 0$. Let

$$v(x) = \sup_{\alpha \in \mathscr{A}_n(x)} J_n(x,\alpha), \quad x \in [0,1] \tag{2}$$

be the value function of the cooperative optimization problem.

When studying the properties of the value function it is convenient to reduce the dimension of the control vector to 1. Recall that the function

$$(g_1 \oplus \dots \oplus g_n)(x) = \inf\{g_1(x_1) + \dots + g_n(x_n) : x_1 + \dots + x_n = x\}$$

is called the *infimal convolution* of $g_1,\dots,g_n$. Let us extend the functions f_i to $\mathbb{R}$ by the values $f_i(u) = -\infty$, $u \notin [0,\overline{\alpha}^i]$ and put

$$\begin{aligned} F(q) &= \sup\{f_1(\alpha_1) + \dots + f_n(\alpha_n) : \alpha_1 + \dots + \alpha_n = q\} \\ &= -((-f_1) \oplus \dots \oplus (-f_n))(q). \end{aligned} \tag{3}$$

The function F is finite on $[0,\overline{q}]$, $\overline{q} = \sum_{i=1}^n \overline{\alpha}^i$, and takes the value $-\infty$ otherwise. From the properties of an infimal convolution it follows that if f_i are continuous (resp., concave), then F is also continuous (resp., concave): see, e.g., [28] (Corollary 2.1 and Theorem 3.1).

Let $q:\mathbb{R}_+\mapsto[0,\overline{q}]$ be a measurable function. Consider the equation

$$X_t^{x,q}=x+\int_0^t b(X_s^{x,q})\,ds-\int_0^t q_s\,ds \tag{4}$$

instead of (1). If $X_t^{x,q}\geqslant 0$, then the strategy q is called admissible. The set of such strategies is denoted by $\mathscr{A}(x)$. Using an appropriate measurable selection theorem (see [27, Theorem 5.3.1]), we conclude that for any $q\in\mathscr{A}(x)$ there exists $\alpha\in\mathscr{A}_n(x)$ such that $F(q_t)=\sum_{i=1}^n f_i(\alpha_t^i)$. It follows that the value function (2) admits the representation

$$v(x)=\sup_{q\in\mathscr{A}(x)} J(x,q),\quad J(x,q)=\int_0^\infty e^{-\beta t}F(q_t)\,dt.$$

Clearly, for any measurable control $q:\mathbb{R}_+\mapsto[0,\overline{q}]$ the trajectory $X^{x,q}$ cannot leave the interval $[0,1]$ through the right boundary. Denote by

$$\tau^{x,q}=\inf\{t\geqslant 0: X_t^{x,q}=0\}$$

the time of population extinction. As usual, we put $\tau^{x,\alpha}=+\infty$ if $X^{x,\alpha}>0$. Note that $q_t=0,\ t\geqslant\tau^{x,q}$ for any admissible control q.

First, we prove directly that v inherits the concavity property of f_i (see Lemma 2 below).

Lemma 1 *Let Y be a continuous solution of the inequality*

$$Y_t\leqslant x+\int_0^t b(Y_s)\,ds-\int_0^t q_s\,ds.$$

Then $Y_t\leqslant X_t^{x,q}$, $t\leqslant\tau:=\inf\{s\geqslant 0: Y_s=0\}$.

Proof We follow [5] (Chapter 1, Theorem 7). Assume that $Y_{t_1}>X_{t_1}^{x,q}$, $t_1\leqslant\tau$. Let $t_0=\max\{t\in[0,t_1]: Y_t\leqslant X_t^{x,q}\}$. We have

$$Y_{t_0}=X_{t_0}^{x,q},\quad Y_t>X_t^{x,q},\quad t\in(t_0,t_1]. \tag{5}$$

The function $Z=Y-X^{x,q}$ satisfies the inequality

$$0\leqslant Z_t\leqslant\int_{t_0}^t (b(Y_s)-b(X_s^{x,q}))\,ds\leqslant K\int_{t_0}^t Z_s\,ds,\quad t\in[t_0,t_1],$$

where K is the Lipschitz constant of b. By the Gronwall inequality (see, e.g., [21, Theorem 1.2.1]) we get a contradiction with (5): $Z_t=0$, $t\in[t_0,t_1]$. □

Lemma 2 *The function v is non-decreasing. If f_i are concave, then v is concave.*

Proof Let $q \in \mathscr{A}(x)$ and $y > x$. Then

$$X_t^{x,q} \leqslant y + \int_0^t b(X_s^{x,q})\,ds - \int_{t_0}^t q_s\,ds.$$

By Lemma 1 we have $X_t^{x,q} \leqslant X_t^{y,q}$ for $t \leqslant \tau^{x,q}$, and hence for all $t \geqslant 0$. It follows that $\mathscr{A}(x) \subset \mathscr{A}(y)$ and $v(x) \leqslant v(y)$.

Let $0 \leqslant x^1 < x^2 \leqslant 1$, $x = \gamma_1 x^1 + \gamma_2 x^2$, $\gamma_1, \gamma_2 > 0$, $\gamma_1 + \gamma_2 = 1$. For $q^i \in \mathscr{A}(x^i)$ by the concavity of b we have

$$\gamma_1 X_t^{x^1,q^1} + \gamma_2 X_t^{x^2,q^2} \leqslant x + \int_0^t b(\gamma_1 X_t^{x^1,q^1} + \gamma_2 X_t^{x^2,q^2})\,dt - \int_0^t (\gamma_1 q_t^1 + \gamma_2 q_t^2)\,dt.$$

Put $q = \gamma_1 q^1 + \gamma_2 q^2$. Applying Lemma 1 to $Y = \gamma_1 X^{x^1,q^1} + \gamma_2 X^{x^2,q^2}$ and $X^{x,q}$ we get the inequality $Y \leqslant X^{x,q}$. It follows that $q \in \mathscr{A}(x)$. By the concavity of F we obtain:

$$J(x,q) \geqslant \int_0^\infty e^{-\beta t}\left(\gamma_1 F(q_t^1) + \gamma_2 F(q_t^2)\right)\,dt = \gamma_1 J(x^1,q^1) + \gamma_2 J(x^2,q^2).$$

It follows that v is concave: $v(x) \geqslant \gamma_1 v(x^1) + \gamma_2 v(x^2)$. □

Let us introduce the Hamiltonian

$$H(x,z) = b(x)z + \widehat{F}(z),$$

$$\widehat{F}(z) = \sup_{q\in[0,\overline{q}]}(F(q) - qz) = \max_{q\in[0,\overline{\alpha}_1+\cdots+\overline{\alpha}_n]} \max\left\{\sum_{i=1}^n f_i(\alpha_i) - zq : \sum_{j=1}^n \alpha_j = q\right\}$$

$$= \sum_{i=1}^n \max_{\alpha_i\in[0,\overline{\alpha}^i]} (f_i(\alpha_i) - z\alpha_i). \quad (6)$$

Recall that a continuous function $w : [0,1] \mapsto \mathbb{R}$ is called a *viscosity subsolution* (resp., a *viscosity supersolution*) of the Hamilton-Jacobi-Bellman (HJB) equation

$$\beta w(x) - H(x, w'(x)) = 0 \quad (7)$$

on a set $K \subset [0,1]$, if for any $x \in K$ and any test function $\varphi \in C^1(\mathbb{R})$ such that x is a local maximum (resp., minimum) point of $w - \varphi$, relative to K, the inequality

$$\beta w(x) - H(x, \varphi'(x)) \leqslant 0 \quad (\text{resp.}, \geqslant 0)$$

holds true. A function $w \in C([0,1])$ is called a *constrained viscosity solution* (see [26]) of (7) if u is a viscosity subsolution on $[0,1]$ and a viscosity supersolution on $(0,1)$.

By Lemma 2 the value function is continuous. Hence, by Theorem 2.1 of [26], we conclude that v is the unique constrained viscosity solution of (7). However, in our case it is possible to give a more simple characterization of v.

Lemma 3 *Assume that f_i are concave. Then v is the unique continuous function on $[0,1]$, with $v(0)=0$, satisfying the HJB equation (7) on $(0,1)$ in the viscosity sense.*

Proof Since the equality $v(0)=0$ follows from the definition of v, we need only to prove that a continuous function w with $w(0)=0$, satisfying the equation (7) on $(0,1)$ in the viscosity sense, is uniquely defined. To do this we simply show that w is a viscosity subsolution of (7) on $[0,1]$ and refer to the cited result of [26].

The inequality

$$0 = \beta w(0) \leqslant H(0,\varphi'(0)) = \widehat{F}(\varphi'(0))$$

is evident (for any $\varphi \in C^1(\mathbb{R})$). Furthermore, in the terminology of [9, Definitions 2 and 4], the point $x=1$ is *irrelevant* and *regular* for the left-hand side of the HJB equation. These properties follow from the fact that $z \mapsto \widehat{F}(z)$ is non-increasing and $b(1)=0$. By the result of [9] (Theorem 2), w automatically satisfies the equation (7) in the viscosity sense on $(0,1]$. □

The subsequent study of the value function strongly relies on its characterization given in Lemma 3. Let

$$\partial w(x) = \{\gamma \in \mathbb{R} : w(y)-w(x) \geqslant \gamma(y-x)\},$$
$$\partial^+ w(x) = \{\gamma \in \mathbb{R} : w(y)-w(x) \leqslant \gamma(y-x)\}$$

be the sub- and superdifferential of a function w. Since $H(x,p)$ is convex in p and satisfies the inequality

$$|H(x,p)-H(y,p)| = |(b(x)-b(y))p| \leqslant K|p||x-y|,$$

by [4, Chapter II, Theorem 5.6] we infer that

$$\beta v(x) - H(x,\gamma) = 0, \quad \gamma \in \partial^+ v(x), \quad x \in (0,1). \tag{8}$$

As a concave function, v is differentiable on a set $G \subset (0,1)$ with a countable complement $(0,1)\backslash G$. Moreover, v' is continuous and non-increasing on G (see [23, Theorem 25.2]). Thus,

$$\beta v(x) - H(x, v'(x)) = 0, \quad x \in G. \tag{9}$$

Denote by δ_*^i the least maximum point of f_i:

$$\delta_*^i = \min\left(\arg\max_{u\in[0,\overline{\alpha}^i]} f_i(u)\right).$$

Let us call a strategy α *static* if it does not depend on t.

Assumption 1 *The static strategy* $\delta_* = (\delta_*^1, \dots, \delta_*^n)$ *is not admissible for any* $x \in [0,1]$. *Equivalently, one can assume that* $\tau^{x,\delta_*} < \infty$, *or*

$$\max_{x\in[0,1]} b(x) < \sum_{i=1}^{n} \delta_*^i.$$

In what follows *we suppose that the Assumption 1 is satisfied* without further stipulation.

Denote by

$$v'_+(x) = \lim_{y\downarrow x} \frac{v(y)-v(x)}{y-x}, \qquad v'_-(x) = \lim_{y\uparrow x} \frac{v(y)-v(x)}{y-x}$$

the right and left derivatives of v. It is well known that $\partial^+ v(x) = [v'_+(x), v'_-(x)]$, $x \in (0,1)$ and the set-valued mapping $x \mapsto \partial^+ v(x)$ is non-increasing:

$$\partial^+ v(x) \geqslant \partial^+ v(y), \quad x < y. \tag{10}$$

For $A, B \subset \mathbb{R}$ we write $A \leqslant B$ if $\xi \leqslant \eta$ for all $\xi \in A$, $\eta \in B$.

Lemma 4 *Assume that* f_i *are concave. Then the function* v' *is strictly decreasing on* G, *and* v *is strictly concave and strictly increasing.*

Proof To prove that v is strictly concave it is enough to show that $x \mapsto \partial^+ v(x)$ is strictly decreasing:

$$\partial^+ v(x) > \partial^+ v(y), \quad x < y$$

(see [14, Chapter D, Proposition 6.1.3]). Assume that $\partial^+ v(x) \cap \partial^+ v(y) \neq \emptyset$, $x < y$. Then the interval (x,y) contains some points $x_1 < y_1$, $x_1, y_1 \in G$ such that $v'(x_1) = v'(y_1)$. From (10) it follows that v' is differentiable on (x_1, y_1) and equals to a constant. Differentiating the HJB equation (9), we get

$$\beta v'(x) = b'(x) v'(x), \quad x \in (x_1, y_1).$$

Since b is strictly concave, the equality $b'(x) = \beta$, $x \in (x_1, y_1)$ is impossible. Thus, $v'(x) = 0$, $x \in (x_1, y_1)$ and

$$\beta v(x) = \widehat{F}(0) = \sum_{i=1}^{n} f(\delta_*^i), \quad x \in (x_1, y_1).$$

An optimal solution $\alpha^* \in \mathscr{A}_n(x)$ of the problem (2) exists (see, e.g., [10, Theorem 1]). If $f_i(\alpha_t^{i,*}) < f_i(\delta_*^i) = \max_{u\in[0,\overline{q}^i]} f_i(u)$ on a set of positive measure for at least one index i, then

$$v(x) = J_n(x, \alpha^*) < \sum_{i=1}^{n} \int_0^\infty e^{-\beta t} f_i(\delta_*^i)\, dt = \frac{1}{\beta} \sum_{i=1}^{n} f_i(\delta_*^i).$$

If $f_i(\alpha_t^{i,*}) = f_i(\delta_*^i)$ a.e., $i = 1,\dots,n$, then $\alpha_t^{i,*} \geqslant \delta_*^i$ a.e. by the definition of δ_*. But this is impossible since the strategy δ_* is not admissible for x and a fortiori so is α^* (see Lemma 1).

The obtained contradiction implies that $\partial^+ v$ is strictly decreasing. Hence, v is strictly concave. In view of Lemma 2 this property implies that v is strictly increasing. □

Denote by $g^*(x) = \sup_{y\in\mathbb{R}}(xy - g(y))$ the Young-Fenchel transform of a function $g : \mathbb{R} \mapsto (-\infty,\infty]$. Recall (see [24, Proposition 11.3]) that for a continuous convex function $g : [a,b] \mapsto \mathbb{R}$ we have

$$\partial g^*(x) = \arg \max_{y\in[a,b]} (xy - g(y)). \tag{11}$$

The next result establishes a connection between the differentiability of the value function and the optimality of static strategies.

Lemma 5 *Let f_i be concave. If the value function v is not differentiable at $x_0 \in (0,1)$, then the static strategy $q_t = b(x_0) \in \mathscr{A}(x_0)$ is optimal, and x_0 is uniquely defined by the "golden rule": $b'(x_0) = \beta$.*

Proof Assume that $v'_-(x_0) > v'_+(x_0)$, $x_0 \in (0,1)$. By (8) we have

$$\beta v(x_0) = b(x_0)\gamma + \widehat{F}(\gamma), \quad \gamma \in (v'_+(x_0), v'_-(x_0)). \tag{12}$$

Since

$$\widehat{F}(z) = \sup_q\{-zq - (-F(q)\} = (-F)^*(-z), \tag{13}$$

by (11), (12) we obtain

$$\{\widehat{F}'(\gamma)\} = \{-b(x_0)\} = -\arg \max_{q\in[0,\overline{q}]} (F(q) - \gamma q), \quad \gamma \in (v'_+(x_0), v'_-(x_0)). \tag{14}$$

Hence, $\widehat{F}(\gamma) = F(b(x_0)) - b(x_0)\gamma$, $\gamma \in (v'_+(x_0), v'_-(x_0))$ and $b(x_0) \in \mathscr{A}(x_0)$ is optimal:

$$\beta v(x_0) = F(b(x_0)) = \beta J(x_0, b(x_0)).$$

Now assume that the static strategy $b(x_0)$ is optimal. Let us apply the relations Pontryagin's maximum principle to the stationary solution $(X_t, q_t) = (x_0, b(x_0))$ of (4). Consider the adjoint equation

$$\dot{\psi}(t) = -b'(x_0)\psi(t) \tag{15}$$

and the basic relation of the Pontryagin maximum principle:

$$\psi^0 e^{-\beta t} F(b(x_0)) = \max_{q\in[0,\overline{q}]} \left(\psi^0 e^{-\beta t} F(q) + (b(x_0) - q)\psi(t)\right). \tag{16}$$

We have $\psi(t) = Ae^{-b'(x_0)t}$ for some $A \in \mathbb{R}$. If $(x_0, b(x_0))$ is an optimal solution, then there exist $\psi^0 \in \mathbb{R}_+$, $A \in \mathbb{R}$ such that $(\psi^0, A) \neq 0$ and the relations (15), (16) hold true: see [3, Theorem 1].

Let us rewrite (15), (16) as follows

$$\psi^0 F(b(x_0)) = \max_{q\in[0,\overline{q}]} \left(\psi^0 F(q) + A(b(x_0)-q)e^{(\beta-b'(x_0))t}\right).$$

Assume that $b'(x_0) \neq \beta$. If $\psi^0 = 0$, then we get a contradiction since $b(x_0) - q$ changes sign on $[0,\overline{q}]$. Thus, we may assume that $\psi^0 = 1$:

$$F(b(x_0)) = Ab(x_0)e^{(\beta-b'(x_0))t} + \max_{q\in[0,\overline{q}]} \left(F(q) - Ae^{(\beta-b'(x_0))t}q\right)$$
$$= H(x_0, z_t), \quad z_t = Ae^{(\beta-b'(x_0))t}. \tag{17}$$

But the equality (17) is impossible, since either $|z_t| \to \infty$ and $H(x_0,z_t) \to +\infty$, $t \to \infty$, or $|z_t| \to 0$ and

$$H(x_0,z_t) \to H(x_0,0) = \widehat{F}(0) = \sum_{i=1}^{n} f_i(\delta^i_*), \quad t \to \infty.$$

In the latter case by (3) and (17) we have

$$F(b(x_0)) = \sum_{i=1}^{n} f_i(\nu_i) = \sum_{i=1}^{n} f_i(\delta^i_*)$$

for some $\nu_i \in [0,\overline{\alpha}^i]$ with $\nu_1 + \cdots + \nu_n = b(x_0)$. From the definition of δ^i_* it then follows that $\nu_i \geqslant \delta^i_*$, $i = 1,\ldots,n$. This is a contradiction, since $\sum_{i=1}^n \delta^i_* \notin \mathscr{A}(x_0)$, and $\sum_{i=1}^n \nu^i = b(x_0)$ should retain this property. □

From the properties of b it follows that either $b'(x) < \beta$, $x \in (0,1)$, or the equation

$$b'(x) = \beta, \quad x \in (0,1) \tag{18}$$

has a unique solution $\widehat{x} \in (0,1)$.

Theorem 1 *Suppose that f_i are concave. Then the value function v is strictly increasing, strictly concave and continuously differentiable on $(0,1)$, except maybe the point $\widehat{x}$. If F is differentiable at $b(\widehat{x})$, then v is continuously differentiable.*

Proof From Lemma 5 it follows that $\widehat{x}$ is the only possible discontinuity point of v. If v is not differentiable at $\widehat{x}$, then the interval $(v'_+(\widehat{x}), v'_-(\widehat{x}))$ is non-empty. But if F is differentiable at $b(\widehat{x})$, then (14) gives a contradiction: $F'(b(\widehat{x})) = \gamma$ for all $\gamma \in (v'_+(x_0), v'_-(x_0))$. □

Note that the assumption, concerning the existence of $F'(b(\widehat{x}))$ is not restrictive. Firstly, F' can have only countably many discontinuity points. Thus, $\widehat{x}$ is not one of these points for all $\beta \in D$, where $(0,\infty)\backslash D$ is countable. Secondly, the formula

$$\partial^+ F(q) = \bigcap_{i=1}^{n} \partial^+ f_i(\alpha^i), \quad \sum_{i=1}^{n} \alpha^i = q, \quad \sum_{i=1}^{n} f_i(\alpha^i) = F(q) \tag{19}$$

(see [14, Chapter D, Corollary 4.5.5]) shows that $F'(b(\widehat{x}))$ exists if any of the functions f_i is differentiable at α^i, satisfying (19).

The next result shows that the static strategy $q = b(\widehat{x})$ is indeed optimal.

Theorem 2 *Assume that f_i are concave. A static strategy $b(x_0) \in \mathscr{A}(x_0)$, $x_0 \in (0,1)$ is optimal if and only if x_0 coincides with the solution $\widehat{x}$ of (18).*

Proof The necessity is proved in Lemma 5. It remains to prove that $b(\widehat{x}) \in \mathscr{A}(\widehat{x})$ is optimal. If v is not differentiable at $\widehat{x}$, the result follows from Lemma 4. Assume that v is continuously differentiable.

The convex function $\widehat{F}$ is continuously differentiable on a co-countable set $U \subset \mathbb{R}$. Furthermore, v is twice differentiable a.e., and $v'' \leqslant 0$ a.e., since v' is decreasing. Hence, $\widehat{F}(v'(x))$ is differentiable on the co-countable set $(v')^{-1}(U) = \{x \in (0,1) : v'(x) \in U\}$. Differentiating the HJB equation (9), by the chain rule we obtain

$$(\beta - b'(x))v'(x) = v''(x)\left(b(x) + \widehat{F}'(v'(x))\right) \quad a.e.$$

The inequalities

$$\beta - b'(x) < 0, \quad x \in (0, \widehat{x}); \quad \beta - b'(x) > 0, \quad x \in (\widehat{x}, 1)$$

imply that $v''(x) < 0$ a.e. and

$$b(x) + \widehat{F}'(v'(x)) > 0, \quad \text{a.e. on } (0, \widehat{x}), \quad b(x) + \widehat{F}'(v'(x)) < 0, \quad \text{a.e. on } (\widehat{x}, 1). \tag{20}$$

Since v' is continuous and strictly decreasing we get the inequalities

$$b(\widehat{x}) + \widehat{F}'_+(v'(\widehat{x})) \geqslant 0 \geqslant b(\widehat{x}) + \widehat{F}'_-(v'(\widehat{x})).$$

Using (11), (13), we obtain

$$b(\widehat{x}) \in -\partial \widehat{F}(v'(\widehat{x})) = \arg\max_{q \in [0, \overline{q}]} \{F(q) - v'(\widehat{x})q\}. \tag{21}$$

It follows that the static strategy $q_t = b(\widehat{x}) \in \mathscr{A}(\widehat{x})$ is optimal:

$$\beta v(\widehat{x}) = b(\widehat{x})v'(\widehat{x}) + \widehat{F}(v'(\widehat{x})) = F(b(\widehat{x})), \quad v(\widehat{x}) = J(\widehat{x}, b(\widehat{x})).$$

□

We turn to the analysis of optimal strategies $q \in \mathscr{A}(x)$ for $x \neq \widehat{x}$. Put

$$\widehat{q}(z) = -\partial \widehat{F}(z). \tag{22}$$

On the co-countable set U, where $\widehat{F}$ is differentiable, the mapping (22) is single-valued. By (21) we have

$$\widehat{q}(v'(x)) = \arg\max_{q \in [0, \overline{q}]} (F(q) - qv'(x)), \quad v'(x) \in U.$$

Note, that $H_z(x,z) = b(x) - \widehat{q}(z)$, $z \in U$. From (20) we know that

$$H_z(x,v'(x)) > 0, \quad \text{a.e. on } (0,\widehat{x}), \qquad H_z(x,v'(x)) < 0, \quad \text{a.e. on } (\widehat{x},1).$$

We want to use $\widehat{q}(v'(x))$ as a *feedback control*, formally considering the equation

$$\dot{X} = b(X) - \widehat{q}(v'(X)) = H_z(X,v'(X)), \quad X_0 = x.$$

To do it in a rigorous way let us first introduce

$$\tau^x = \int_x^{\widehat{x}} \frac{du}{H_z(u,v'(u))}.$$

This definition allows τ^x to be infinite. Let $x < \widehat{x}$ (resp., $x > \widehat{x}$). Then the mapping

$$\Psi(y) = \int_x^{y} \frac{du}{H_z(u,v'(u))}, \quad \Psi : (x,\widehat{x}) \mapsto (0,\tau^x) \quad (\text{resp.}, \Psi : (\widehat{x},x) \mapsto (0,\tau^x))$$

is a bijection.

Lemma 6 *Let $\psi : [a,b] \mapsto \mathbb{R}$ be continuous and strictly monotonic. Then ψ^{-1} is absolutely continuous if and only if $\psi' \neq 0$ a.e. on (a,b).*

By Lemma 6, which proof can be found in [29] (Theorem 2), the equation

$$t = \int_x^{Y_t} \frac{du}{H_z(u,v'(u))} \tag{23}$$

uniquely defines a locally absolutely continuous function Y_t, $t \in (0,\tau^x)$. Moreover, Y is strictly increasing if $x < \widehat{x}$ and strictly decreasing if $x > \widehat{x}$. From (23) we get

$$\dot{Y}_t = H_z(Y_t,v'(Y_t)) = b(Y_t) - \widehat{q}(v'(Y_t)) \quad \text{a.e. on } (0,\tau^x), \quad Y_0 = x. \tag{24}$$

Theorem 3 *Let f_i be concave and $x \neq \widehat{x}$. Put $\mathscr{T} = \{t \in (0,\tau^x) : v'(Y_t) \in U\}$, where Y is defined by* (23). *Define the strategy*

$$q_t^* = \widehat{q}(v'(Y_t)), \quad t \in \mathscr{T}.$$

On the countable set $(0,\tau^x)\backslash\mathscr{T}$ the values q_t^* can be defined in an arbitrary way. If τ^x is finite put

$$q_t^* = b(\widehat{x}), \quad t \geqslant \tau^x.$$

The strategy $q^* \in \mathscr{A}(x)$ is optimal.

Proof The equality (24) means that $Y_t = X^{x,q^*}$ on $(0,\tau^x)$. Furthermore, $X^{x,q^*} = \widehat{x}$ on $[\tau^x,\infty)$ by the definition of q^*. Clearly, q^* is admissible. To prove that q^* is optimal it is enough to show that

$$W_t = \int_0^t e^{-\beta s} F(q_s^*)\,ds + e^{-\beta t} v(X_t^{x,q^*})$$

is constant, since then

$$W_0 = v(x) = \lim_{t\to\infty} W_t = \int_0^\infty e^{-\beta s} F(q_s^*)\,ds.$$

We have

$$\begin{aligned}\dot{W}_t &= e^{-\beta t} F(q_t^*) + e^{-\beta t}\left(-\beta v(X_t^{x,q^*}) + v'(X_t^{x,q^*})(b(X_t^{x,q^*}) - q_t^*)\right)\\ &= e^{-\beta t}(-\beta v(X_t^{x,q^*}) + H(X_t^{x,q^*}, v'(X_t^{x,q^*}))) = 0 \quad \text{a.e. on } (0,\tau^x).\end{aligned}$$

For $t > \tau^x$ we have

$$\begin{aligned}W_t &= \int_0^\tau e^{-\beta s} F(q_s^*)\,ds + \frac{F(b(\widehat{x}))}{\beta}(e^{-\beta\tau} - e^{-\beta t}) + e^{-\beta t} v(\widehat{x})\\ &= \int_0^\tau e^{-\beta s} F(q_s^*)\,ds + \frac{F(b(\widehat{x}))}{\beta} e^{-\beta\tau},\end{aligned}$$

since $v(\widehat{x}) = F(b(\widehat{x}))/\beta$ by the optimality of the static strategy $b(\widehat{x})$. □

From Theorem 3 we see that if the solution $\widehat{x}$ of (18) exists, then it attracts any optimal trajectory. Moreover, X^{x,q^*} is strictly increasing (resp., decreasing) on $(0,\tau^x)$, if $x < \widehat{x}$ (resp. $x > \widehat{x}$).

We also mention that the multivalued feedback control $\widehat{q}(v'(x))$ satisfies the inequalities

$$b(x) > \widehat{q}(v'(x)), \quad x \in (0,\widehat{x}); \quad b(x) < \widehat{q}(v'(x)), \quad x \in (\widehat{x},1). \tag{25}$$

Indeed, $\widehat{q}(z) = -\partial F(z)$ is a non-increasing multivalued mapping. On a co-countable set U the mappings $\widehat{q}(v'(x))$ are single-valued, non-decreasing and satisfy the inequalities (20). Thus, in any neighbourhood of a point $x \neq \widehat{x}$ there exist $x_1 < x$, $x_2 > x$ such that

$$\widehat{q}(v'(x_1)) \leqslant \widehat{q}(v'(x)) \leqslant \widehat{q}(v'(x_2)),$$

where $\widehat{q}(v'(x_i))$ are single-valued and satisfy (20). It easily follows that

$$b(x) \geqslant \widehat{q}(v'(x)), \quad x \in (0,\widehat{x}); \quad b(x) \leqslant \widehat{q}(v'(x)), \quad x \in (\widehat{x},1). \tag{26}$$

Assume that $b(x_0) \in \widehat{q}(v'(x_0))$, $x_0 \neq \widehat{x}$. Then from the HJB equation (9) it follows that $q = b(x_0) \in \mathscr{A}(x_0)$ is an optimal strategy: $\beta v(x_0) = F(b(x_0))$, in contradiction with Lemma 5. Thus, the inequalities (26) are strict.

3. Cooperative harvesting problem: the case of non-concave revenues

Now we drop the assumption that f_i are concave. Let us extend the class of harvesting strategies. A family $(\mu_t(dx))_{t\geqslant 0}$ of probability measures on $[0,\overline{q}]$ is called a *relaxed control* if the function

$$t \mapsto \int_0^{\overline{q}} \varphi(y)\,\mu_t(dy)$$

is measurable for any continuous function φ. A relaxed control μ induces the dynamics

$$X_t = x + \int_0^t b(X_s)\,ds - \int_0^t \int_0^{\overline{q}} y\mu_s(dy)\,ds.$$

The related value function is defined as follows

$$v_r(x) = \sup_{\mu\in\mathscr{A}^r(x)} J^r(x,\mu), \quad J^r(x,\mu) = \int_0^\infty e^{-\beta t} \int_0^{\overline{q}} F(y)\mu_t(dy)\,dt, \quad x\in[0,1], \tag{27}$$

where $\mathscr{A}^r = \{\mu : X^{x,\mu} \geqslant 0\}$ is the class of admissible relaxed controls.

Denote by $\widetilde{F}$ the concave hull of F: $\widetilde{F} = -(-F)^{**}$. Let

$$\widetilde{v}(x) = \sup_{q\in\mathscr{A}(x)} \widetilde{J}(x,q), \quad \widetilde{J}(x,q) = \int_0^\infty e^{-\beta t}\widetilde{F}(q_t)\,dt \tag{28}$$

be the related value function. Note that by (3) and the properties of infimal convolution ([16], Chapter 3, § 3.4, Theorem 1) we have

$$-\widetilde{F} = (-F)^{**} = (-f_1)^{**} \oplus \cdots \oplus (-f_n)^{**} = (-\widetilde{f}_1) \oplus \cdots \oplus (-\widetilde{f}_n),$$

where $\widetilde{f}_i$ and f^{**} are the convex hull and the double Young-Fenchel transformation of f respectively. Hence,

$$\widetilde{F}(q) = \sup\{\widetilde{f}_1(\alpha_1) + \cdots + \widetilde{f}_n(\alpha_n) : \alpha_1 + \cdots + \alpha_n = q\}. \tag{29}$$

Since $\widetilde{F} \geqslant F$ it follows that $\widetilde{v} \geqslant v$. By the Jensen inequality we have

$$J^r(x,\mu) \leqslant \int_0^\infty e^{-\beta t} \int_0^{\overline{q}} \widetilde{F}(y)\mu_t(dy)\,dt \leqslant \int_0^\infty e^{-\beta t}\widetilde{F}(\widetilde{q}_t)\,dt,$$

where $q_t = \int_0^{\overline{q}} y\mu_t(dy)$ is an admissible control for the problem (4). Thus,

$$v(x) \leqslant v_r(x) \leqslant \widetilde{v}(x).$$

Lemma 7 *For any $p \in [0, \overline{q}]$ there exists $p_1, p_2 \in [0, \overline{q}]$, $\varkappa \in (0,1)$ such that*

$$p = \varkappa p_1 + (1-\varkappa) p_2, \quad \widetilde{F}(p) = \varkappa F(p_1) + (1-\varkappa) F(p_2).$$

The proof of a more general result can be found in [14] (Chapter E, Proposition 1.3.9(ii)).

Denote by $\widetilde{q}_t$ the strategy, constructed in Theorem 3, where F is replaced by $\widetilde{F}$. We claim that

$$\widetilde{F}(\widetilde{q}_t) = F(\widetilde{q}_t), \quad \text{a.e. on } (0, \tau^x). \tag{30}$$

By construction, $\widetilde{q}_t$ is the unique maximum point of $q \mapsto \widetilde{F}(q) - qv'(Y_t)$ on $[0, \overline{q}]$ for all $t \in \widetilde{\mathscr{T}}$, where $(0, \tau^x) \backslash \widetilde{\mathscr{T}}$ is countable. If $\widetilde{F}(\widetilde{q}_t) \neq F(\widetilde{q}_t)$, $t \in \widetilde{\mathscr{T}}$ then, by Lemma 7, $\widetilde{F}$ is affine in an open neighbourhood of $\widetilde{q}_t$, and

$$\arg \max_{q \in [0, \overline{q}]} (\widetilde{F}(q) - v'(Y_t) q\}$$

contains this neighbourhood: a contradiction.

Furthermore, by Lemma 7 there exist $p_1, p_2 \in [0,1]$, $\varkappa \in (0,1)$ such that

$$b(\widehat{x}) = \varkappa p_1 + (1-\varkappa) p_2, \qquad \widetilde{F}(b(\widehat{x})) = \varkappa F(p_1) + (1-\varkappa) F(p_2). \tag{31}$$

Consider the static relaxed control

$$\mu_s = \begin{cases} \widetilde{q}_s, & s < \tau^x, \\ \varkappa \delta_{p_1} + (1-\varkappa) \delta_{p_2}, & s \geqslant \tau^x, \end{cases} \tag{32}$$

where δ_a is the Dirac measure, concentrated at a. By (30), (31) we have

$$J^r(x, \mu) = \int_0^{\tau^x} e^{-\beta t} F(\widetilde{q}_t)\, dt + \int_{\tau^x}^{\infty} e^{-\beta t} (\varkappa F(p_1) + (1-\varkappa) F(p_2))\, dt = \widetilde{J}(x, \widetilde{q}).$$

Thus, $v_r(x) = \widetilde{v}(x)$ and the strategy (32) is optimal for the relaxed problem (27).

To prove that $v_r(x) = v(x)$ let us construct an approximately optimal strategy

$$q^\varepsilon \in \mathscr{A}(x) : J(x, q^\varepsilon) \to v_r(x), \quad \varepsilon \to 0. \tag{33}$$

We may assume that $p_1 \neq p_2$ and $p_1 < b(\widehat{x}) < p_2$. Otherwise, the strategy (32) reduces to an ordinary control $\mu_s = \widetilde{q}_s I_{\{s < \tau^x\}} + b(\widehat{x}) I_{\{s \geqslant \tau^x\}}$ and we conclude that $v(x) = v_r(x) = \widetilde{v}(x)$.

Define g by the equation

$$\int_{\widehat{x}-\varepsilon}^{\widehat{x}} (b(\widehat{x}) - b(x)) \rho(x)\, dx = \int_{\widehat{x}}^{\widehat{x}+g(\varepsilon)} (b(x) - b(\widehat{x})) \rho(x)\, dx, \tag{34}$$

$$\rho(x) = \frac{1}{(b(x) - p_1)(p_2 - b(x))}.$$

Note, that for sufficiently small $\varepsilon > 0$ we have $\rho(x) > 0$ on $(\widehat{x}-\varepsilon, g(\varepsilon))$ and integrands in (34) are positive. Clearly, $g(\varepsilon) \downarrow 0$, $\varepsilon \to 0$. Put

$$\tau_1 = \int_{\widehat{x}}^{\widehat{x}+g(\varepsilon)} \frac{dx}{b(x)-p_1}, \quad \tau_2 = \int_{\widehat{x}-\varepsilon}^{\widehat{x}+g(\varepsilon)} \frac{dx}{p_2-b(x)},$$

$$\tau_3 = \int_{\widehat{x}-\varepsilon}^{\widehat{x}} \frac{dx}{b(x)-p_1}, \quad \tau = \tau_1+\tau_2+\tau_3.$$

For brevity, we omit the dependence of τ_i on ε. Put

$$q_t^{\varepsilon} = \sum_{j=0}^{\infty} \left(p_1 I_{[j\tau, j\tau+\tau_1)}(t) + p_2 I_{[j\tau+\tau_1, j\tau+\tau_1+\tau_2)}(t) + p_1 I_{[j\tau+\tau_1+\tau_2,(j+1)\tau)}(t) \right). \tag{35}$$

The trajectory $X^{\widehat{x},q^{\varepsilon}}$ is periodic:

$$\begin{aligned}
\dot{X}_t^{\widehat{x},q^{\varepsilon}} &= b(X_t^{\widehat{x},q^{\varepsilon}}) - p_1, \quad (j\tau, j\tau+\tau_1), \quad X_{j\tau}^{\widehat{x},q^{\varepsilon}} = \widehat{x}, \\
\dot{X}_t^{\widehat{x},q^{\varepsilon}} &= b(X_t^{\widehat{x},q^{\varepsilon}}) - p_2, \quad (j\tau+\tau_1, j\tau+\tau_1+\tau_2), \quad X_{j\tau+\tau_1}^{\widehat{x},q^{\varepsilon}} = \widehat{x}+g^{\varepsilon}, \\
\dot{X}_t^{\widehat{x},q^{\varepsilon}} &= b(X_t^{\widehat{x},q^{\varepsilon}}) - p_1, \quad (j\tau+\tau_1+\tau_2, (j+1)\tau)), \quad X_{j\tau+\tau_1+\tau_2}^{\widehat{x},q^{\varepsilon}} = \widehat{x}-\varepsilon.
\end{aligned}$$

It sequentially visits the points $\widehat{x}, \widehat{x}+g^{\varepsilon}, \widehat{x}-\varepsilon, \widehat{x}$ and moves monotonically between them. Furthermore,

$$\int_{j\tau}^{(j+1)\tau} e^{-\beta t} F(q_t^{\varepsilon})\,dt = \frac{e^{-\beta j\tau}}{\beta} \Big((1-e^{-\beta\tau_1})F(p_1) + (e^{-\beta\tau_1} - e^{-\beta(\tau_1+\tau_2)})F(p_2)$$
$$+ (e^{-\beta(\tau_1+\tau_2)} - e^{-\beta\tau})F(p_1) \Big)$$

Thus,

$$J(\widehat{x}, q^{\varepsilon}) = \frac{1}{\beta(1-e^{-\beta\tau})} \Big((1-e^{-\beta\tau_1})F(p_1) + (e^{-\beta\tau_1} - e^{-\beta(\tau_1+\tau_2)})F(p_2)$$
$$+ (e^{-\beta(\tau_1+\tau_2)} - e^{-\beta\tau})F(p_1) \Big) = \frac{1}{\beta} \left(\frac{\tau_1+\tau_3}{\tau} F(p_1) + \frac{\tau_2}{\tau} F(p_2) \right) + o(1), \quad \varepsilon \to 0.$$

Since

$$\tau_1 = \frac{g(\varepsilon)}{b(\widehat{x})-p_1}(1+o(1)), \quad \tau_2 = \frac{g(\varepsilon)+\varepsilon}{p_2-b(\widehat{x})}(1+o(1)), \quad \tau_3 = \frac{\varepsilon}{b(\widehat{x})-p_1}(1+o(1)),$$

using (31), we get

$$\frac{\tau_1+\tau_3}{\tau_2} = \frac{p_2-b(\widehat{x})}{b(\widehat{x})-p_1} = \frac{\varkappa}{1-\varkappa},$$

$$\frac{\tau_1+\tau_3}{\tau} = \frac{1}{1+\tau_2/(\tau_1+\tau_3)} = \varkappa, \qquad \frac{\tau_2}{\tau} = \frac{1}{1+(\tau_1+\tau_3)/\tau_2} = 1-\varkappa.$$

Thus,

$$\lim_{\varepsilon\to 0} J(\widehat{x}, q^{\varepsilon}) = \frac{1}{\beta}(\varkappa F(p_1)+(1-\varkappa)F(p_2)) = \frac{\widetilde{F}(b(\widehat{x}))}{\beta} = v(\widehat{x}).$$

We see that the strategy (35) satisfies (33), and $v(x) = v_r(x) = v(x)$. The obtained results are summarized below.

Theorem 4 *The value functions* (2), (27), (28) *coincide:* $v = v_r = \widetilde{v}$. *By Theorem* 1, *applied to* (28)*,* v *is strictly increasing, strictly concave and continuously differentiable on* $(0,1)$*, except maybe the point* $\widehat{x}$*. If* $\widetilde{F}$ *is differentiable at* $b(\widehat{x})$*, then* v *is continuously differentiable. The strategy* (32) *is optimal for the relaxed problem* (27)*.*

4. Rational taxation

Assume that a regulator imposes the proportional tax $v'(x)\alpha$ for the fishing intensity α. Then the myopic agents take their optimal strategies from the sets

$$\widehat{\alpha}^i(x) = \arg\max_{u\in[0,\overline{\alpha}^i]}\{f_i(u) - v'(x)u\}.$$

The direct implementation of such feedback controls may cause technical problems, since the related equation (1) can be unsolvable. Instead of continuous change of the tax $v'(X_t)$, a more realistic approach consists in its fixing for some periods of time: $v'(X_{\tau_j})$, $t\in[\tau_j,\tau_{j+1})$. In this case agents also fix their strategies:

$$\alpha^i_{\tau_i} \in \arg\max_{u\in[0,\overline{\alpha}^i]}\{f_i(u) - v'(X_{\tau_j})u\}, \quad t\in[\tau_j,\tau_{j+1}).$$

This scheme results in "step-by-step positional control" (see [18]), defined recursively by the formulas:

$$\begin{aligned} X_0^{x,\alpha} &= x, \\ \alpha^i_t = \alpha^i_{\tau_j} &\in \arg\max_{u\in[0,\overline{\alpha}^i]}\{f_i(u) - v'(X_{\tau_j}^{x,\alpha})u\}, \quad t\in[\tau_j,\tau_{j+1}), \qquad (36)\\ X_t^{x,\alpha} &= X_{\tau_j}^{x,\alpha} + \int_{\tau_j}^{t} b(X_s^{x,\alpha})\,ds - \sum_{i=1}^{n}\alpha^i_{\tau_j}\cdot(t-\tau_j), \quad t\in[\tau_j,\tau_{j+1}), \\ 0 &= \tau_0 < \ldots \tau_j < \ldots, \quad \tau_j\to\infty, \quad j\to\infty, \qquad (37) \end{aligned}$$

bypassing at the same time the mentioned technical problems.

Theorem 5 *Let $\widetilde{F}'(\widehat{x})$ exist. Then for any $\varepsilon > 0$, $\delta > 0$ there exists a sequence* (37) *such that the strategy* (36) *is approximately optimal: $J_n(x,\alpha) \geqslant v(x) - \varepsilon$ and stabilizing in the following sense:*

$$|X_t^{x,\alpha} - \widehat{x}| < \delta, \quad t \geqslant \bar{t}(x,\varepsilon,\delta).$$

Proof First note that

$$\widehat{\alpha}^i(z) := \arg \max_{u\in[0,\overline{\alpha}^i]} (f_i(u) - zu) \subset \widetilde{\alpha}^i(z) := \arg \max_{u\in[0,\overline{\alpha}^i]} (\widetilde{f}_i(u) - zu).$$

Indeed, if $u^* \in \widehat{\alpha}^i(z)$, then $-z \in \partial(-f_i)(u^*)$ and $u^* \in \partial(-f_i)^*(-z)$: see [14, Chapter E, Proposition 1.4.3]. But, by (11),

$$\partial(-f_i)^*(-z) = \arg \max_{u\in[0,\overline{\alpha}^i]} (-zu - (-f_i)^{**}(u)) = \arg \max_{u\in[0,\overline{\alpha}^i]} (\widetilde{f}_i(u) - zu) = \widetilde{\alpha}^i(z).$$

Furthermore, from the representation (29) we get

$$\max_{q\in[0,\overline{q}]} \{\widetilde{F}(q) - zq\} = \sum_{i=1}^n \max_{\alpha_i\in[0,\overline{\alpha}^i]} \{\widetilde{f}_i(\alpha_i) - z\alpha_i\}$$

(see also (6)). Thus,

$$\widetilde{q}(z) := \arg \max_{q\in[0,\overline{q}]} (\widetilde{F}(q) - zq) = \sum_{i=1}^n \widetilde{\alpha}^i(z) \supset \sum_{i=1}^n \widehat{\alpha}^i(z). \tag{38}$$

From (25) it then follows that

$$\begin{aligned} b(x) &> \sum_{i=1}^n \widehat{\alpha}^i(v'(x)), \quad x \in (0,\widehat{x}), \\ b(x) &< \sum_{i=1}^n \widehat{\alpha}^i(v'(x)), \quad x \in (\widehat{x},1). \end{aligned} \tag{39}$$

The subsequent argumentation follows the introductory section of [17]. For any $x_0 \in (0,1)$ and any $\alpha_0^i \in \widehat{\alpha}^i(v'(x_0))$ we have

$$\beta v(x_0) = \left(b(x_0) - \sum_{i=1}^n \alpha_0^i \right) v'(x_0) + \sum_{i=1}^n f_i(\alpha_0^i).$$

Put,

$$\psi(x,\alpha) = -\beta v(x) + \left(b(x) - \sum_{i=1}^n \alpha^i \right) v'(x) + \sum_{i=1}^n f_i(\alpha^i)$$

and define the time moment

$$\tau_1 = \inf\{t \geqslant 0 : \psi(X_t^{x_0,\alpha_0}, \alpha_0) < -\beta\varepsilon \text{ or } X_t^{x_0,\alpha_0} > \widehat{x} + \delta\}, \quad x_0 \in (0, \widehat{x}), \tag{40}$$

$$\tau_1 = \inf\{t \geqslant 0 : \psi(X_t^{x_0,\alpha_0}, \alpha_0) < -\beta\varepsilon \text{ or } X_t^{x_0,\alpha_0} < \widehat{x} - \delta\}, \quad x_0 \in (\widehat{x}, 1), \tag{41}$$

$$\tau_1 = \inf\{t \geqslant 0 : \psi(X_t^{x_0,\alpha_0}, \alpha_0) < -\beta\varepsilon \text{ or } X_t^{x_0,\alpha_0} \notin (\widehat{x} - \delta, \widehat{x} + \delta)\}, \quad x_0 = \widehat{x}. \tag{42}$$

For $t \in [0, \tau_1]$ in each of the cases (40), (41), (42) we have respectively

$$X_t^{x_0,\alpha_0} \in [x_0, \widehat{x} + \delta], \quad X_t^{x_0,\alpha_0} \in [\widehat{x} - \delta, x_0], \quad X_t^{x_0,\alpha_0} \in [\widehat{x} - \delta, \widehat{x} + \delta].$$

Assume that x_{k-1}, α_{k-1}, τ_k are defined. Put

$$x_k = X_{\tau_k}^{x_{k-1},\alpha_{k-1}}, \quad \alpha_k^i \in \widehat{\alpha}^i(v'(x_k)),$$

$$\tau_{k+1} = \inf\{t \geqslant \tau_k : \psi(X_t^{x_k,\alpha_k}, \alpha_k) < -\beta\varepsilon \text{ or } X_t^{x_k,\alpha_k} > \widehat{x} + \delta\}, \quad x_k \in (0, \widehat{x}), \tag{43}$$

$$\tau_{k+1} = \inf\{t \geqslant \tau_k : \psi(X_t^{x_k,\alpha_k}, \alpha_k) < -\beta\varepsilon \text{ or } X_t^{x_k,\alpha_k} < \widehat{x} - \delta\}, \quad x_k \in (\widehat{x}, 1), \tag{44}$$

$$\tau_{k+1} = \inf\{t \geqslant \tau_k : \psi(X_t^{x_k,\alpha_k}, \alpha_k) < -\beta\varepsilon \text{ or } X_t^{x_k,\alpha_k} \notin (\widehat{x} - \delta, \widehat{x} + \delta)\}, \quad x_k = \widehat{x}. \tag{45}$$

The function $x \mapsto \psi(x, \alpha)$ is uniformly continuous on any interval $[a, b] \subset (0, 1)$ uniformly in $\alpha \in [0, \overline{q}]$. Thus, there exists δ' such that if

$$|\psi(x, \alpha) - \psi(y, \alpha)| \geqslant \beta\varepsilon, \quad [x, y] \subset [a, b],$$

then $|x - y| \geqslant \delta'$. Assume that $\psi(X_{\tau_{k+1}}^{x_k,\alpha_k}, \alpha_k) = -\beta\varepsilon$. Since $\psi(x_k, \alpha_k) = 0$, we get

$$\delta' \leqslant |X_{\tau_{k+1}}^{x_k,\alpha_k} - x_k| \leqslant \int_{\tau_k}^{\tau_{k+1}} b(X_t^{x_k,\alpha_k})\, dt + \int_{\tau_k}^{\tau_{k+1}} \sum_{i=1}^{n} \alpha_k^i\, dt \leqslant (\overline{b} + \overline{q})(\tau_{k+1} - \tau_k),$$

where $\overline{b} = \max_{x \in [0,1]} b(x)$. Furthermore, if $\psi(X_{\tau_{k+1}}^{x_k,\alpha_k}) > -\beta\varepsilon$ and $\tau_{k+1} < \infty$, then in any of three cases (43), (44), (45) we have

$$\delta \leqslant |X_{\tau_{k+1}}^{x_k,\alpha_k} - x_k| \leqslant (\overline{b} + \overline{q})(\tau_{k+1} - \tau_k).$$

Thus, the differences $\tau_{k+1} - \tau_k$ are uniformly bounded from below by a positive constant, and the strategy $\alpha = \sum_{k=0}^{\infty} \alpha_k I_{[\tau_k, \tau_{k+1})}(t)$ is well defined for all $t \geqslant 0$. Note, that $X_t^{x_0,\alpha}$ belongs to one of the sets $[x_0, \widehat{x} + \delta]$, $[\widehat{x} - \delta, x_0]$, $[\widehat{x} - \delta, \widehat{x} + \delta]$ for all $t \geqslant 0$.

By the Berge maximum theorem (see [1, Theorem 17.31]) the set-valued mapping $\widehat{\alpha}$ is upper hemicontinuous, hence its graph is closed (see [1, Theorem 17.10]). From (39) it then follows that there is a finite gap between $b(x)$ and $\sum_{i=1}^{n} \widehat{\alpha}^i(v'(x))$ on $(0, \widehat{x} - \delta) \cup (\widehat{x} + \delta, 1)$. Thus, $|\dot{X}^{\alpha,x_0}|$ is uniformly bounded from below by a positive constant, when $X^{\alpha,x_0} \in (0, \widehat{x} - \delta) \cup (\widehat{x} + \delta, 1)$. This property implies that X^{α,x_0} reaches the neighbourhood $[\widehat{x} - \delta, \widehat{x} + \delta]$ in finite time $\overline{t}(x, \varepsilon, \delta)$. After reaching this neighbourhood, X^{α,x_0} remains in it forever by the construction of α.

It remains to prove that α is ε-optimal. We have

$$-\beta v(X_t^{x_k,\alpha_k}) + \left(b(X_t^{x_k,\alpha_k}) - \sum_{i=1}^{n} \alpha_k^i\right) v'(X_t^{x_k,\alpha_k}) + \sum_{i=1}^{n} f_i(\alpha_k^i) \geqslant -\beta\varepsilon, \quad t \in (\tau_k, \tau_{k+1}).$$

After the multiplication on $e^{-\beta t}$ an integration we get

$$e^{-\beta\tau_{k+1}} v(X_{\tau_{k+1}}^{x_k,\alpha_k}) - e^{-\beta\tau_k} v(X_{\tau_k}^{x_k,\alpha_k}) + \int_{\tau_k}^{\tau_{k+1}} e^{-\beta t} \sum_{i=1}^{n} f_i(\alpha_k^i)\, dt \geqslant \varepsilon(e^{-\beta\tau_{k+1}} - e^{-\beta\tau_k}).$$

Summing up and passing to the limit we obtain the desired inequality:

$$\int_0^\infty e^{-\beta t} \sum_{i=1}^{n} f_i(\alpha_t^i)\, dt \geqslant v(x_0) - \varepsilon.$$

□

As an example, consider the problem with n identical agents and assume that their common profit function is linear: $f_i(u) = f(u) = u$, $u \in [0, \overline{\alpha}]$. The HJB equation (9) takes the form

$$\beta v(x) = b(x)v'(x) + n \max_{u \in [0,\overline{\alpha}]} (u - v'(x)u).$$

From (21) it follows that $v'(\widehat{x}) = 1$. Thus,

$$v'(x) > 1, \quad x < \widehat{x}, \quad v'(x) < 1, \quad x > \widehat{x} \tag{46}$$

and v satisfies the equations

$$\beta v(x) = b(x)v'(x), \quad x < \widehat{x}; \qquad \beta v(x) = (b(x) - n\overline{\alpha})v'(x) + n\overline{\alpha}, \quad x > \widehat{x}.$$

Solving these equations, by the uniqueness result, given in Lemma 3, we infer that

$$v(x) = \frac{b(\widehat{x})}{\beta} \exp\left(-\int_x^{\widehat{x}} \frac{\beta}{b(y)}\, dy\right), \quad x \in (0, \widehat{x}],$$

$$v(x) = \frac{1}{\beta}(b(\widehat{x}) - n\overline{\alpha}) \exp\left(\int_{\widehat{x}}^{x} \frac{\beta}{b(y) - \overline{\alpha}n}\, dy\right) + \frac{1}{\beta} n\overline{\alpha}, \quad x \in [\widehat{x}, 1].$$

For the biomass quantities x below the critical level $\widehat{x}$ the tax $v'(x)$ does not depend on n:

$$v'(x) = \frac{b(\widehat{x})}{b(x)} \exp\left(-\int_x^{\widehat{x}} \frac{\beta}{b(y)}\, dy\right), \quad x \in (0, \widehat{x}].$$

For larger values of x we have

$$v'(x) = \frac{n\overline{\alpha} - b(\widehat{x})}{n\overline{\alpha} - b(x)} \exp\left(-\int_{\widehat{x}}^{x} \frac{\beta}{n\overline{\alpha} - b(y)} dy \right), \quad x \in [\widehat{x}, 1].$$

In particular, $v'(x) \to f'(0) = 1$, $n \to \infty$.

Note, that a tax, stimulating an optimal cooperative behavior is by no means unique. For instance, any tax, satisfying (46), can serve this purpose. So, the most interesting quantity is the "critical tax"

$$v'(\widehat{x}) = \widetilde{F}'(b(\widehat{x})). \tag{47}$$

The equality (47) follows from (21). Consider $\widetilde{F}$ as the value function of the elementary problem (29), where the artificial agents with concave revenues $\widetilde{f}_i$ cooperatively distribute some given harvesting intensity q. Formula (47) shows that $v'(\widehat{x})$ is simply the shadow price of the critical growth growth rate $b(\widehat{x})$ within this problem.

We are interested in the dependence of the critical tax $v'(\widehat{x})$ on the size of agent community. Consider again n identical agents with the revenue functions $f_i = f$. If f is linear, the critical tax, as we have seen, does not depend on n. Assume now that f is differentiable and strictly concave. Then by (21) and (38) we get

$$b(\widehat{x}) \in \sum_{i=1}^{n} \arg \max_{u \in [0,\overline{\alpha}]} \{f(u) - v'(\widehat{x})u\}$$

Taking optimal values of u to be equal, we conclude that $v'(\widehat{x}) = f'(b(\widehat{x})/n)$. Thus, $v'(\widehat{x})$ is increasing in n, and $v'(\widehat{x}) \to f'(0)$, $n \to \infty$. Our final result shows that this situation is typical: the critical tax can only increase, when the agent community widens.

Theorem 6 *Denote by F_n, F_{n+m} and v_n, v_{n+m} the cooperative instantaneous revenue functions* (3) *and the value functions* (2)*, corresponding to the agent communities*

$$\{f_i\}_{i=1}^{n} \subset \{f_i\}_{i=1}^{n+m}.$$

Assume that $\widetilde{F}_n'(b(\widehat{x}))$, $\widetilde{F}_{n+m}'(b(\widehat{x}))$ exist. Then

$$v_n'(\widehat{x}) = \widetilde{F}_n'(b(\widehat{x})) \leqslant v_{n+m}'(\widehat{x}) = \widetilde{F}_{n+m}'(b(\widehat{x})).$$

Proof It is enough to consider the case $m = 1$. By the associativity of the infimal convolution we have

$$(-\widetilde{F}_{n+1})(q) = (-\widetilde{F}_n) \oplus (-\widetilde{f}_{n+1})(q).$$

The formula for the subdifferential of an infimal convolution, given in [14, Chapter D, Corollary 4.5.5], implies that

$$\partial(-\widetilde{F}_{n+1})(q) \subseteq \bigcup_{u} \partial(-\widetilde{F}_n)(u) \cap \partial(-\widetilde{f}_{n+1})(q-u) \subseteq \bigcup_{u \in [0,q]} \partial(-\widetilde{F}_n)(u).$$

But since the set-valued mapping $u \mapsto \partial(-\widetilde{F}_{n+1})(u)$ is non-decreasing, we have

$$\partial(-\widetilde{F}_{n+1})(q) \leqslant \partial(-\widetilde{F}_n)(q), \quad q \in [0,\overline{q}].$$

Thus, $\widetilde{F}'_{n+1}(b(\widehat{x})) \geqslant \widetilde{F}'_n(b(\widehat{x}))$. □

A resembling result for discrete time problem was proved in [25, Theorem 3].

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