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Analytic solutions of transcendental equations with application to automatics

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In the paper the extremal dynamic error $x(\tau)$ and the moment of time τ are considered. The extremal value of dynamic error gives information about accuracy of the system. The time τ gives information about velocity of transient. The analytical formulae enable design of the system with prescribed properties. These formulae are calculated due to the assumption that $x(\tau)$ is a function of the roots s_1, ..., s_n of the characteristic equation.

Key words: extremal problems, characteristic equation, transmittance.

1. Introduction

In the paper [1] the necessary conditions for the extremal value $x(\tau)$ are presented. In the article [2] the method of the decomposition of nth order system into a set of 2-nd order systems is given. In this article some new results are obtained.

2. Statement of the problem

Calculation of conditions and extremum of the extreme value of the dynamic error [1]. Let us consider the differential equation determining the dynamic error in a linear control system of nth order with lumped and constant parameters:

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_{n-1} \frac{dx}{dt} + a_n x = 0. \tag{1}$$

The initial conditions are determined by the force function and the system's parameters. Let us assume in general, that

$$x^{(i)}(0) = c_{i+1} \neq \; 0 \text{ for } \; i = 0, 1, ..., n-1$$

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We assume further that the characteristic equation of equation (1) has m different real roots and $2p$ different complex roots.

It is evident that

$$m+2p=n.$$

We denote by s_k real roots and

$$\alpha_k+j\omega_k=r_k,\quad \alpha_k-j\omega_k=\hat{r}_k\ (k=1,2,...,p).$$

The solution of equation (1) takes the form

$$x(t)=\sum_{k=1}^{m}A_k e^{s_k t}+\sum_{k=1}^{p}\left[B_k\cos\left(\omega_k t\right)+C_k\sin\left(\omega_k t\right)\right]e^{\alpha_k t} \tag{2}$$

where $A_k,\ B_k,\ C_k,\ s_k,\ \alpha_k$ and ω_k are real numbers.

The necessary conditions for the dynamic error $x(t)$ to attain an extreme value at $t=\tau$ is given by the relation:

$$\begin{aligned}\frac{dx}{dt}=&\sum_{k=1}^{m}A_k s_k e^{s_k\tau}+\\&+\sum_{k=1}^{p}\left[(-B_k\sin\omega_k\tau+C_k\cos\omega_k\tau)\omega_k+(B_k\cos\omega_k\tau+C_k\sin\omega_k\tau)\alpha_k\right]e^{\alpha_k\tau}=0.\end{aligned} \tag{3}$$

The constants are determined from

$$x^{(i)}(0)=c_{i+1}=\sum_{k=1}^{m}A_k s_k^i+\sum_{k=1}^{p}\left[B_k\,\mathrm{Re}(r_k^i)+C_k\,\mathrm{Im}(r_k^i)\right]\ (i=0,1,...,n-1). \tag{4}$$

The extreme value of the dynamic error is

$$x(\tau)=\sum_{k=1}^{m}A_k e^{s_k\tau}+\sum_{k=1}^{p}\left[B_k\cos\left(\omega_k\tau\right)+C_k\sin\left(\omega_k\tau\right)\right]e^{\alpha_k\tau}. \tag{5}$$

The extremum of extreme value of the dynamic error given by equation (5), computed with regard to the parameters $s_k,\ \alpha_k,\ \omega_k$, is obtained by putting the respective partial derivatives of $x(\tau)$ equal to zero.

Denoting by

$$\left(\frac{\partial x(\tau)}{\partial s_k}\right)^*,\ \left(\frac{\partial x(\tau)}{\partial \alpha_k}\right)^*,\ \left(\frac{\partial x(\tau)}{\partial \omega_k}\right)^*$$

the partial derivatives of expression (5) for constant τ we may write

$$\left.\begin{aligned}\frac{\partial x(\tau)}{\partial s_k} &= \left(\frac{\partial x(\tau)}{\partial s_k}\right)^* + \frac{\partial x(\tau)}{\partial \tau}\frac{\partial \tau}{\partial s_k}\\ \frac{\partial x(\tau)}{\partial \alpha_k} &= \left(\frac{\partial x(\tau)}{\partial \alpha_k}\right)^* + \frac{\partial x(\tau)}{\partial \tau}\frac{\partial \tau}{\partial \alpha_k}\\ \frac{\partial x(\tau)}{\partial \omega_k} &= \left(\frac{\partial x(\tau)}{\partial \omega_k}\right)^* + \frac{\partial x(\tau)}{\partial \tau}\frac{\partial \tau}{\partial \omega_k}\end{aligned}\right\}. \tag{6}$$

However, we have from equation (3)

$$\frac{\partial x(\tau)}{\partial \tau} = 0$$

and therefore

$$\left.\begin{aligned}\frac{\partial x(\tau)}{\partial s_k} &= \left(\frac{\partial x(\tau)}{\partial s_k}\right)^*\\ \frac{\partial x(\tau)}{\partial \alpha_k} &= \left(\frac{\partial x(\tau)}{\partial \alpha_k}\right)^*\\ \frac{\partial x(\tau)}{\partial \omega_k} &= \left(\frac{\partial x(\tau)}{\partial \omega_k}\right)^*\end{aligned}\right\}. \tag{7}$$

We obtain the following conditions:

$$\left.\begin{aligned}&\sum_{k=1}^{m}\frac{\partial A_k}{\partial s_j}e^{s_k\tau} + A_j\tau e^{s_j\tau} + \sum_{k=1}^{p}\left(\frac{\partial B_k}{\partial s_j}\cos\omega_k\tau + \frac{\partial C_k}{\partial s_j}\sin\omega_k\tau\right)e^{\alpha_k\tau} = 0\\ &\hspace{20em} j = 1,2,\ldots,m\\ &\sum_{k=1}^{m}\frac{\partial A_k}{\partial \alpha_j}e^{s_k\tau} +\\ &+\sum_{k=1}^{p}\left(\frac{\partial B_k}{\partial \alpha_j}\cos\omega_k\tau + \frac{\partial C_k}{\partial \alpha_j}\sin\omega_k\tau\right)e^{\alpha_k\tau} + (B_j\cos\omega_j\tau + C_j\sin\omega_j\tau)\,e^{\alpha_j\tau}\tau = 0\\ &\sum_{k=1}^{m}\frac{\partial A_k}{\partial \omega_j}e^{s_k\tau} +\\ &+\sum_{k=1}^{p}\left(\frac{\partial B_k}{\partial \omega_j}\cos\omega_k\tau + \frac{\partial C_k}{\partial \omega_j}\sin\omega_k\tau\right)e^{\alpha_k\tau} + (C_j\cos\omega_j\tau - B_j\sin\omega_j\tau)\,e^{\alpha_j\tau}\tau = 0\\ &\hspace{20em} j = 1,2,\ldots,p\end{aligned}\right\} \tag{8}$$

In this way we have a system of n linear and homogenous equations with n unknowns $e^{s_k\tau}$, $e^{\alpha_k\tau}\sin\omega_k\tau$, $e^{\alpha_k\tau}\cos\omega_k\tau$.

The determinant of system (8) must vanish if there are not to be all zero solutions. The same determinant (after being reflected about one of the main diagonals) is:

$$|D+A\tau| \tag{9}$$

where D and A are matrices determined by the following equations:

$$\left.\begin{aligned}
D = {} & \sum_{j=1}^{m}\sum_{k=1}^{m}\frac{\partial A_j}{\partial s_k}E_{jk} + \sum_{j=1}^{p}\sum_{k=1}^{m}\left(\frac{\partial B_j}{\partial s_k}E_{m+2j-1,k} + \frac{\partial C_j}{\partial s_k}E_{m+2j,k}\right) + \\
& + \sum_{j=1}^{m}\sum_{k=1}^{p}\left(\frac{\partial A_j}{\partial \alpha_k}E_{j,m+2k-1} + \frac{\partial A_j}{\partial \omega_k}E_{j,m+2k}\right) + \\
& + \sum_{j=1}^{p}\sum_{k=1}^{p}\left[\left(\frac{\partial B_j}{\partial \alpha_k}E_{m+2j-1,m+2k-1} + \frac{\partial B_j}{\partial \omega_k}E_{m+2j-1,m+2k}\right) + \right. \\
& \left. + \left(\frac{\partial C_j}{\partial \alpha_k}E_{m+2j,m+2k-1} + \frac{\partial C_j}{\partial \omega_k}E_{m+2j,m+2k}\right)\right], \\
A = {} & \sum_{j=1}^{m}A_jE_{jj} + \sum_{j=1}^{p}[B_j(E_{m+2j-1,m+2j-1} - E_{m+2j,m+2j}) + \\
& + C_j(E_{m+2j-1,m+2j} + E_{m+2j,m+2j-1})]
\end{aligned}\right\}, \tag{10}$$

$$\begin{aligned}
& E_{jk} = \left(e_{\mu,\nu}^{(jk)}\right) \quad \mu,\nu = 1,\ldots,n \\
& e_{\mu,p}^{(jk)} = \delta_{\mu j}\delta_{\nu k} = \begin{cases} 1 & \text{for } \mu=j,\nu=k \\ 0 & \text{for all other cases.} \end{cases}
\end{aligned} \tag{11}$$

Finally we have

$$|D+A\tau| = 0 \tag{12}$$

and system (8) yields for unknown τ (after some algebraic manipulations) the following equation:

$$(-1)^n\tau^n\prod_{k=1}^{m}A_k\prod_{k=1}^{p}\left(B_k^2+C_k^2\right) = 0. \tag{13}$$

From relation (13) it is evident that the necessary conditions for $x[\tau(s_1,s_2,\ldots,s_n)]$ to have an extremum with respect to $(s_1,s_2,\ldots,s_n)$ are

$$\tau = 0 \tag{14}$$

which means that

$$c_2 = 0 \tag{15}$$

or

$$A_k = 0 \ \text{ for } \ k = 1,2,\ldots,m \tag{16}$$

or

$$(B_k^2 + C_k^2) = 0 \text{ for } k = 1, 2, \ldots, p. \tag{17}$$

By applying Laplace transformation to the equation (1) we obtain the transform $X(s)$ of the error $x(t)$.

$$X(s) = \frac{s^{n-1}c_1 + (a_1c_1 + c_2)s^{n-2} + (a_2c_1 + a_1c_2 + c_3)s^{n-3} + \cdots + (a_{n-1}c_1 + a_{n-2}c_2 + \ldots + a_1c_{n-1} + c_n)}{s^n + a_1s^{n-1} + a_2s^{n-2} + \cdots + a_{n-1}s + a_n} = \frac{L(s)}{M(s)}. \tag{18}$$

The coefficients A_k, B_k, C_k as we know, are equal

$$A_k = \frac{L(s_k)}{\left[\frac{dM(s)}{ds}\right]_{s=s_k}} \quad k = 1, 2, \ldots, m \tag{19}$$

and they can attain zero value if the Sylvester's determinant of the polynomials $L(s)$ and $M(s)$ is equal zero. We obtain the following theorem:

Theorem 2 *The vanishing of the coefficients A_k (16) or $(B_k^2 + C_k^2)$ (17) is possible if the numerator $L(s)$ and the denominator $M(s)$ of the transform $X(s)$ have a common root, it means $A_k = 0$ if $L(s_k) = 0$ and $M(s_k) = 0$.*

In order to eliminate the root s_k which will satisfy both equations $L(s) = 0$ and $M(s) = 0$ we can also use Euclides algorithm [2].

The division of $M(s)$ by $L(s)$ gives

$$s_1 = \frac{c_2}{c_1} \tag{20}$$

for arbitrary natural power $n = 1, 2, \ldots$

In the particular cases, which are often in practice, $c_2 = 0$, then from (20) we obtain $s_1 = 0$ [4]. This value is not useful, because this is limit of stability of (1). In this paper these particular cases, using another method, are considered for $c_2 = 0$, $c_1 \neq 0$, $c_3 \neq 0$ [3].

3. Solution of the problem

Let us consider the equation for $n = 3$

$$x(t) = \sum_{i=1}^{3} A_i e^{s_i t} \tag{21}$$

with the initial conditions: $x(0) = c_1$, $x^{(1)}(0) = c_2$, $x^{(2)}(0) = c_3$. The characteristic equation is

$$s^3 + a_1 s^2 + a_2 s + a_3 = 0 \tag{22}$$

where a_1, a_2, $a_3 > 0$ and $a_1 a_2 - a_3 > 0$, are conditions for stability. Let us assume that roots of the equation (22) fulfill the relation

$$s_3 = \frac{s_1 + s_2}{2} \quad \text{and} \quad s_1 < s_3 < s_2. \tag{23}$$

This assumption denotes that localization of the roots can satisfy one of the three possibilities (Fig. 1). The roots of equation (22) are determined by the relation (23) and Vieta's formulae

$$a_1 = -(s_1 + s_2 + s_3) = -3s_3$$

then

$$s_3 = -\frac{1}{3}a_1. \tag{24}$$

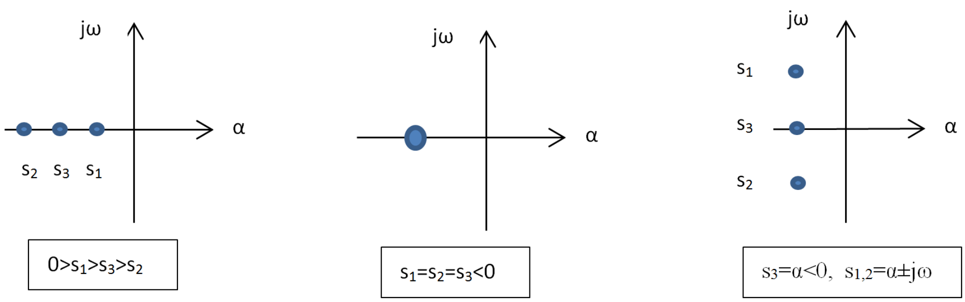

Figure 1: Localization of roots.

The division of the equation (22) by the root s_3 gives

$$s^2 + \frac{2}{3}a_1 s + a_2 - \frac{2}{9}a_1^2 = 0 \tag{25}$$

and

$$a_3 = \frac{1}{3}a_1 a_2 - \frac{2}{27}a_1^3. \tag{26}$$

The roots of equation (25) are

$$s_{1,2} = -\frac{1}{3}a_1 \pm \sqrt{\frac{1}{3}a_1^2 - a_2}. \tag{27}$$

From the relation (27) we deduce:

1^0 If $\frac{1}{3}a_1^2 - a_2 > 0$ there are three real, different roots $s_1 \neq s_2 \neq s_3$ (24) and (27),

2^0 If $\frac{1}{3}a_1^2 - a_2 = 0$ there is one triple root

$$s_1 = -\frac{1}{3}a_1 \tag{28}$$

3^0 If $\frac{1}{3}a_1^2 - a_2 < 0$ there is one real root $s_3 = -\frac{1}{3}a_1$,

$$s_3 = -\frac{1}{3}a_1 \tag{29}$$

and two complex roots

$$s_{1,2} = \alpha \pm j\omega \tag{30}$$

where

$$\left.\begin{aligned} \alpha &= -\frac{1}{3}a_1 \\ \omega &= \sqrt{a_2 - \frac{1}{3}a_1^2} \end{aligned}\right\}. \tag{31}$$

The coefficients of equation (21) are

$$\left.\begin{aligned} A_1 &= \frac{c_3 - c_2(s_2+s_3) + c_1 s_2 s_3}{(s_1-s_2)(s_1-s_3)} \\ A_2 &= \frac{c_3 - c_2(s_1+s_3) + c_1 s_1 s_3}{(s_1-s_2)(s_3-s_2)} \\ A_3 &= \frac{c_3 - c_2(s_1+s_2) + c_1 s_1 s_2}{(s_1-s_3)(s_2-s_3)} \end{aligned}\right\}. \tag{32}$$

From the necessary condition for extremum $x(t)$ determined by (21) we have

$$\frac{dx}{dt} = s_1 A_1 e^{s_1\tau} + s_2 A_2 e^{s_2\tau} + s_3 A_3 e^{s_3\tau} = 0. \tag{33}$$

After division of (33) by $e^{s_3\tau}$ we obtain

$$s_1 A_1 e^{(s_1-s_3)\tau} + s_2 A_2 e^{(s_2-s_3)\tau} + s_3 A_3 = 0. \tag{34}$$

From the relation (23) we obtain a very important relation

$$s_1 - s_3 = -(s_2 - s_3) \tag{35}$$

which substituted to (34) gives

$$s_1 A_1 e^{(s_1-s_3)\tau} + s_2 A_2 e^{-(s_1-s_3)\tau} + s_3 A_3 = 0. \tag{36}$$

After multiplying (36) by $e^{(s_1-s_3)\tau}$ we obtain finally

$$s_1 A_1 e^{2(s_1-s_3)\tau} + s_3 A_3 e^{(s_1-s_3)\tau} + s_2 A_2 = 0. \tag{37}$$

It is quadratic equation with respect to $e^{(s_1-s_3)\tau}$, then we have:

In the case 1^0, that is of real $s_1 \neq s_2 \neq s_3$ and $\frac{1}{3}a_1^2 - a_2 > 0$

$$\tau_{1,2} = \frac{1}{s_1 - s_3} \ln \left(-\frac{1}{2}\frac{s_3 A_3}{s_1 A_1} \pm \sqrt{\left(\frac{1}{2}\frac{s_3 A_3}{s_1 A_1}\right)^2 - \frac{s_2 A_2}{s_1 A_1}} \right) \tag{38}$$

or

$$\tau_{1,2} = \frac{1}{\sqrt{\frac{1}{3}a_1^2 - a_2}} \ln \left(-\frac{1}{2}\frac{s_3 A_3}{s_1 A_1} \pm \sqrt{\left(\frac{1}{2}\frac{s_3 A_3}{s_1 A_1}\right)^2 - \frac{s_2 A_2}{s_1 A_1}} \right). \tag{39}$$

If

$$\left(\frac{1}{2}\frac{s_3 A_3}{s_1 A_1}\right)^2 < \frac{s_2 A_2}{s_1 A_1} \tag{40}$$

then no extremum exist. If

$$\left.\begin{aligned} \left(\frac{1}{2}\frac{s_3 A_3}{s_1 A_1}\right)^2 &= \frac{s_2 A_2}{s_1 A_1} \\ -\frac{1}{2}\frac{s_3 A_3}{s_1 A_1} &> 1 \end{aligned}\right\} \tag{41}$$

then we have one extremum. If

$$\left.\begin{aligned} \left(\frac{1}{2}\frac{s_3 A_3}{s_1 A_1}\right)^2 &> \frac{s_2 A_2}{s_1 A_1} \\ -\frac{1}{2}\frac{s_3 A_3}{s_1 A_1} &\geqslant 1 \end{aligned}\right\} \tag{42}$$

then we have two extremums.

In the case 2^0, $s_1 = s_2 = s_3$ the moment of time τ is determined by the equation

$$x^{(1)}(\tau) = [s_1 A_3 \tau^2 + (s_1 A_2 + 2A_3)\tau + (s_1 A_1 + A_2)]e^{s_1 \tau} = 0 \tag{43}$$

and coefficients A_k are

$$A_k = \sum_{i=1}^{k} \frac{x^{(k-i)}(0)(-1)^{i-1}s^{i-1}}{(i-1)!(k-i+1)!}, \quad k = 0, 1, 2, \ldots, n. \tag{44}$$

For $n = 3$ we have

$$\left.\begin{aligned} s &= -\frac{1}{3}a_1 \\ A_1 &= c_1 \\ A_2 &= (c_2 - s c_1) \\ A_3 &= \frac{1}{2}c_3 - s c_2 - \frac{1}{2}s^2 c_1 \end{aligned}\right\}. \tag{45}$$

From the equation (43) we have

$$\tau = \frac{-(sA_2+2A_3) \pm \sqrt{(sA_2+2A_3)^2 - 4sA_3(sA_1+A_2)}}{2sA_3}. \tag{46}$$

Similarly to the case 1^0 one or two extremums may exist or no extremum exists.

In the case 3^0:

$$\left.\begin{aligned} s_3 &= \alpha \\ s_1 &= \alpha + j\omega \\ s_2 &= \alpha - j\omega \end{aligned}\right\}. \tag{47}$$

where

$$\left.\begin{aligned} \alpha &= -\frac{1}{3}a_1 \\ \omega &= \sqrt{a_2 - \frac{1}{3}a_1^2} \end{aligned}\right\}. \tag{48}$$

The equation (34) takes a form

$$\alpha A_3 e^{\alpha\tau} + [A_1 cos\omega\tau + A_2 sin\omega\tau]e^{\alpha\tau} = 0$$

or after division by $e^{\alpha\tau}$ we obtain

$$\alpha A_3 + [A_1 cos\omega\tau + A_2 sin\omega\tau] = 0. \tag{49}$$

From this equation we conclude that there may be infinitely many extremums. In order to obtain the explicit relations of time τ with the initial conditions c_i $(i = 1,2,3)$ and roots s_1, s_2, s_3 we take into account the relations between coefficients A_1, A_2, A_3 with the initial conditions and the roots [5],[6].

In the paper [3] it was proved that the necessary and sufficient condition for the positive and aperiodic solutions of equation (21) is

1^0

$$c_2 = 0, \;\; c_1 > 0, \tag{50}$$

2^0 the roots s_1, s_2, s_3 are real and negative,

3^0

$$\left.\begin{aligned} \frac{c_3}{c_1} &= -s_1 s_2 \\ &\text{or} \\ \frac{c_3}{c_1} &= -s_1 s_3 \\ &\text{or} \\ \frac{c_3}{c_1} &= -s_2 s_3. \end{aligned}\right\}. \tag{51}$$

The solution of equation (21) has only one extremum at the moment $\tau = 0$, because $c_2 = \frac{dx}{dt}\Big|_{t=0} = 0$ by the assumption.

In Fig. 2,3,4 there are presented solutions of (5) for the different $\frac{c_3}{c_1}$ and s_1, s_2, s_3 which fulfill the conditions (50) and (51).

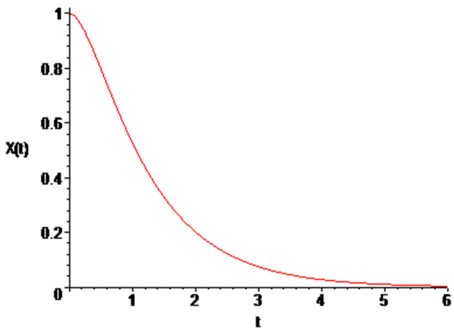

Figure 2: Transient of the error for: $s_1 = -1$, $s_2 = -3$, $s_3 = -2$, $c_2 = 0$, $c_3/c_1 = -3$.

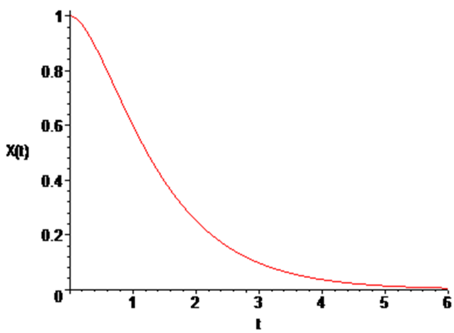

Figure 3: Transient of the error for: $s_1 = -1$, $s_2 = -3$, $s_3 = -2$, $c_2 = 0$, $c_3/c_1 = -2$.

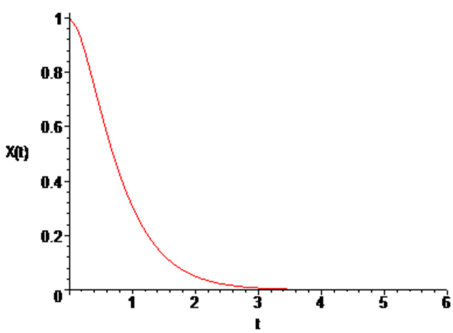

Figure 4: Transient of the error for: $s_1 = -1$, $s_2 = -3$, $s_3 = -2$, $c_2 = 0$, $c_3/c_1 = -6$.

In the considerations that follow we assume that

$$s_3 = \frac{s_1 + s_2}{2} \tag{52}$$

and that

$$s_2 < s_3 < s_1 < 0. \tag{53}$$

The solution of equation (21) is

$$x(t) = \frac{[s_2(s_2+s_1)c_1 - (3s_2+s_1)c_2 + 2c_3]e^{s_1 t}}{(s_1-s_2)^2} + \frac{[s_1(s_1+s_2)c_1 - (3s_1+s_2)c_2 + 2c_3]e^{s_2 t}}{(s_1-s_2)^2} + \\ + \frac{[-4s_1s_2c_1 + 4(s_1+s_2)c_2 - 4c_3]e^{\left(\frac{s_1+s_2}{2}t\right)}}{(s_1-s_2)^2}. \tag{54}$$

The derivative of $x(t)$ is

$$\frac{dx(t)}{dt} = \frac{[s_1s_2(s_2+s_1)c_1 - s_1(3s_2+s_1)c_2 + 2s_1c_3]e^{s_1 t}}{(s_1-s_2)^2} \\ + \frac{[s_1s_2(s_1+s_2)c_1 - s_2(3s_1+s_2)c_2 + 2s_2c_3]e^{s_2 t}}{(s_1-s_2)^2} \\ + \frac{[-2s_1s_2(s_1+s_2)c_1 + 2(s_1+s_2)^2c_2 - 2(s_1+s_2)c_3]e^{\left(\frac{s_1+s_2}{2}\right)t}}{(s_1-s_2)^2}. \tag{55}$$

Using the necessary condition for extremum $\left.\frac{dx(t)}{dt}\right|_{t=\tau} = 0$ we obtain from (55)

$$\tau_1 = \frac{-2}{s_1 - s_2} \ln\left(\frac{[-s_1s_2(s_1+s_2)c_1 + (s_1+s_2)^2c_2 - (s_1+s_2)c_3] + \sqrt{A}}{s_2[-s_1(s_1+s_2)c_1 + (3s_1+s_2)c_2 - 2c_3]}\right), \tag{56}$$

$$\tau_2 = \frac{-2}{s_1 - s_2} \ln\left(\frac{[-s_1s_2(s_1+s_2)c_1 + (s_1+s_2)^2c_2 - (s_1+s_2)c_3] - \sqrt{A}}{s_2[-s_1(s_1+s_2)c_1 + (3s_1+s_2)c_2 - 2c_3]}\right), \tag{57}$$

where

$$A = -(s_1-s_2)^2(-c_2^2s_2^2 + s_2^2s_1c_1c_2 + 2s_2c_2c_3 - 3c_2^2s_1s_2 + s_2s_1^2c_1c_2 + 2c_3s_1c_2 - s_1^2c_2^2 - c_3^2).$$

In Fig. 5 the plot of the solution $x(t)$ is presented for $c_1 = 1$, $c_2 = 1$, $c_3 = 3$ and the roots $s_1 = -1$, $s_2 = -3$, $s_3 = -2$.

$$x(t) = 7e^{-t} + 4e^{-3t} - 10e^{-2t}.$$

The extremum is only one at the moment of time $\tau = 0.69314718$.

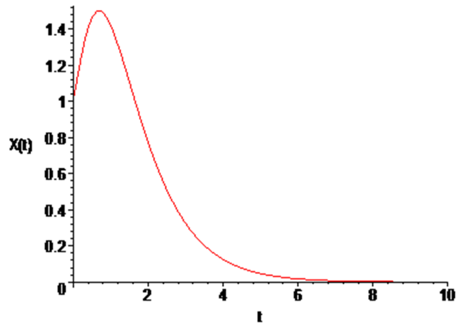

Figure 5: Transient of the error.

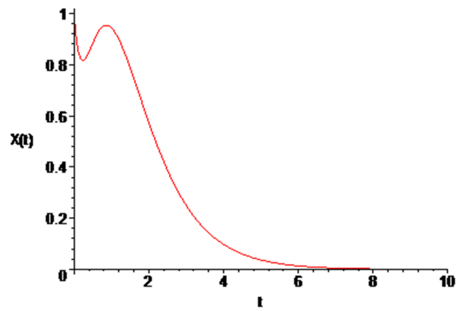

Figure 6: Transient of the error.

In the next example, where $c_1 = 1$, $c_2 = -2$, $c_3 = 15$ we have two extremums at times $\tau_1 = 0.2352329$ and $\tau_2 = 0.863379$. The solution $x(t)$ is presented in Fig. 6.

$$x(t) = \frac{11}{2}e^{-t} + \frac{11}{2}e^{-3t} - 10e^{-2t}.$$

In the particular case when $c_2 = 0$ the general formulae (56) and (57) are simpler

$$\begin{aligned}\tau_1 &= \frac{-2}{s_1 - s_2} \ln \left(\frac{s_1 s_2^2 c_1 + c_3 s_2 + s_1^2 s_2 c_1 + c_3 s_1 + \sqrt{(s_1 - s_2)^2 c_3^2}}{s_2(s_1^2 c_1 + s_1 s_2 c_1 + 2c_3)} \right) \\ &= \frac{-2}{s_1 - s_2} \ln \left(\frac{s_1(s_2^2 c_1 + s_1 s_2 c_1 + 2c_3)}{s_2(s_1^2 c_1 + s_1 s_2 c_1 + 2c_3)} \right)\end{aligned} \tag{58}$$

$$\begin{aligned}\tau_2 &= \frac{-2}{s_1 - s_2} \ln \left(\frac{s_1 s_2^2 c_1 + c_3 s_2 + s_1^2 s_2 c_1 + c_3 s_1 - \sqrt{(s_1 - s_2)^2 c_3^2}}{s_2(s_1^2 c_1 + s_1 s_2 c_1 + 2c_3)} \right) \\ &= \frac{-2}{s_1 - s_2} \ln \left(\frac{s_2(s_1^2 c_1 + s_1 s_2 c_1 + 2c_3)}{s_2(s_1^2 c_1 + s_1 s_2 c_1 + 2c_3)} \right) = 0.\end{aligned} \tag{59}$$

The condition $c_2 = 0$ gives minimal transient error (see [1]). In this particular case when $c_2 = 0$ and $c_1 = 1$, $c_3 = 3$ extremum is at the moment $\tau_1 = 0$, because $c_2 = 0$, and at the moment of time $\tau_2 = 0.510825$ (Fig.7) and

$$x(t) = \frac{9}{2}e^{-t} + \frac{5}{2}e^{-3t} - 6e^{-2t}.$$

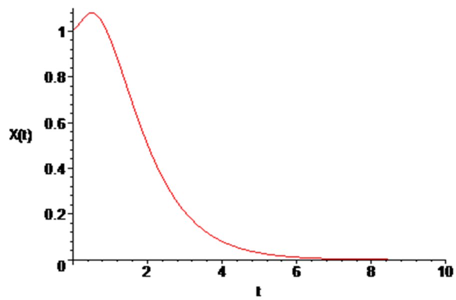

Figure 7: Transient of the error.

The case 2^0

$$s_1 = s_2 = s_3 = -\frac{1}{3}a_1. \tag{60}$$

The solution is

$$x(t) = \left[\left(\frac{1}{2}c_3 + \frac{1}{2}s_1^2c_1 - s_1c_2\right)t^2 + (c_2 - s_1c_1)t + c_1\right]e^{s_1t}. \tag{61}$$

The derivative

$$\frac{dx(t)}{dt} = \left[\left(\frac{1}{2}s_1c_3 + \frac{1}{2}s_1^3c_1 - s_1^2c_2\right)t^2 + (c_3 - s_1c_2)t + c_2\right]e^{s_1t}. \tag{62}$$

From the condition that $\frac{dx(t)}{dt} = 0$ we obtain

$$\tau_1 = \frac{-c_3 + s_1c_2 + \sqrt{c_3^2 - 4c_3c_2s_1 + 5s_1^2c_2^2 - 2c_2c_1s_1^3}}{s_1c_3 + s_1^3c_1 - 2s_1^2c_2}, \tag{63}$$

$$\tau_2 = \frac{-c_3 + s_1c_2 - \sqrt{c_3^2 - 4c_3c_2s_1 + 5s_1^2c_2^2 - 2c_2c_1s_1^3}}{s_1c_3 + s_1^3c_1 - 2s_1^2c_2}. \tag{64}$$

For numerical example according to $s_1 = -1$, $c_1 = 1$, $c_2 = 1$, $c_3 = 2$ the solution of $x(t)$ is presented in Fig. 8.

$$x(t) = \left(\frac{5}{2}t^2 + 2t + 1\right)e^{-t}$$

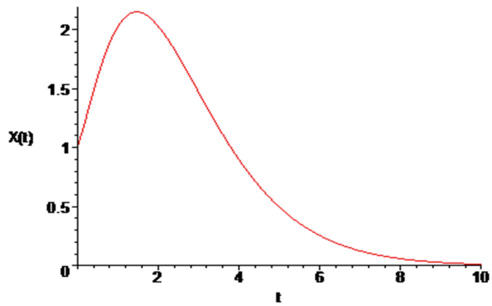

Figure 8: Transient of the error.

and from (64) we have $\tau = 1.471779$.

In the case when $c_2 = 0$ the formulae (63) and (64) are

$$\tau_1 = \frac{1}{2}\left(\frac{-2c_3 + 2\sqrt{c_3^2}}{s_1c_3 + s_1^3c_1}\right) = 0, \tag{65}$$

$$\tau_2 = \frac{-2c_3}{s_1(c_3 + s_1^2c_1)}. \tag{66}$$

<u>The case 3^0</u>

$$s_3 = \alpha,\;\; s_1 = \alpha + j\omega,\;\; s_2 = \alpha - j\omega,\;\; c_1 \neq 0,\;\; c_2 = 0,\;\; c_3 \neq 0. \tag{67}$$

The solution is

$$x(t) = -\frac{e^{\alpha t}[-c_1\alpha^2 - c_1\omega^2 - c_3 + \cos(t\omega)c_1\alpha^2 + c_3\cos(t\omega) + \alpha c_1\omega\sin(t\omega)}{\omega^2}. \tag{68}$$

The derivative

$$\frac{dx(t)}{dt} = -\frac{e^{\alpha t}[-c_1\alpha^3 - \alpha c_1\omega^2 - \alpha c_3 + \alpha^3c_1\cos(t\omega) + \alpha c_1\omega^2\cos(t\omega) + \alpha c_3\cos(t\omega) - c_3\omega\sin(t\omega)]}{\omega^2} \tag{69}$$

or in a more convenient form

$$\frac{dx(t)}{dt} = \left[\frac{c_3 sin(t\omega)}{\omega} + \frac{(c_1\alpha^3 + \alpha c_1\omega^2 + \alpha c_3)[1 - cos(t\omega)]}{\omega^2}\right]e^{\alpha t}. \tag{70}$$

From the necessary condition $\frac{dx(t)}{dt} = 0$ using (70) we obtain

$$\tau_1 = 2\frac{k\pi}{\omega},\quad k = 0, 1, \ldots, \tag{71}$$

$$\tau_2 = \frac{-2}{\omega}\operatorname{arctg}\left(\frac{c_3\omega}{\alpha(c_1\alpha^2 + c_1\omega^2 + c_3)}\right) + \frac{2k\pi}{\omega}, \quad k = 0, 1, \ldots \tag{72}$$

where for $\alpha = -\frac{1}{3}a_1$, $\omega = \sqrt{a_2 - \frac{1}{3}a_1^2}$ we obtain finally from (72)

$$\tau_2 = 6\frac{-\operatorname{arctg}\left(9\frac{c_3\sqrt{(9a_2-3a_1^2}}{a_1(-9c_3+2c_1a_1^2-9c_1a_2)}\right) + k\pi}{\sqrt{9a_2 - 3a_1^2}}. \tag{73}$$

A numerical example for $\alpha = -1$, $\omega = 2$, $c_1 = 1$, $c_2 = 0$, $c_3 = -3$ is presented in Fig. 9 and the solution is

$$x(t) = \frac{1}{2}[1 + \cos(2t) + \sin(2t)]e^{-t}$$

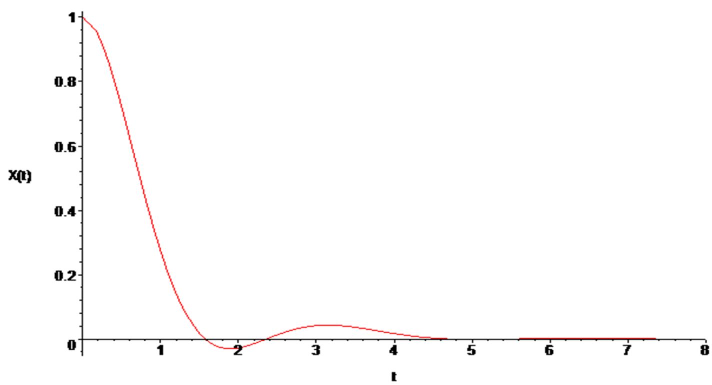

Figure 9: Transient of the error.

Extremums from (71) and (72) are as follows: $\tau_1 = 0$, $\tau_2 = 1.89254688$, $\tau_3 = 3.14159265$, $\tau_4 = 5.03413953$, $\tau_5 = 6.2831853$.

For $c_2 = 0$, the example presented in the paper [2] can be described by the following relations:

Transform of the error

$$E(s) = \frac{1 + 2\xi T_3 s + T_3^2 s^2}{K + (1 + KT_2)s + 2\xi T_3 s^2 + T_3^2 s^3}.$$

The initial conditions are

$$c_1 = 1, \quad c_2 = 0, \quad c_3 = -\frac{KT_2}{T_3^2}.$$

For numerical values: $\xi = 0.75$, $T_3 = 0.1$, $K = 10$, $T_2 = 0.15$ we obtain

$$X(s) = \frac{1 + 0.15s + 0.01s^2}{10 + 2.5s + 0.15s^2 + 0.01s^3}.$$

Solution

$$x(t) = [0.2857142857 + 0.7142857143\cos(13.22875656t) + \\ +0.3779644731\sin(13.22875656t)]e^{-5t}$$

is presented in Fig. 10.

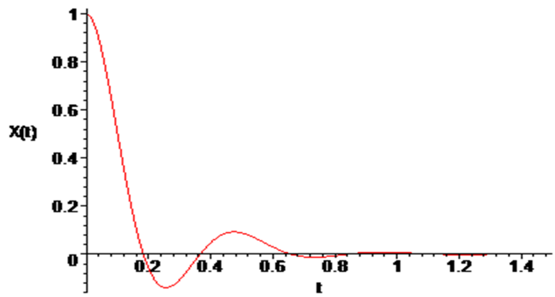

Figure 10: Transient of the error.

From (71) and (72) the extremums of $x(t)$ are at the moments of time $\tau_1 = 0$, $\tau_2 = 0.2564298695$, $\tau_3 = 0.4749641647$, $\tau_4 = 0.7313940343$, $\tau_5 = 0.9499283295$ and so on. The values of the extremum are: $x(\tau_1) = 1$, $x(\tau_2) = -0.13872$, $x(\tau_3) = 0.093031$, $x(\tau_4) = -0.012905$, $x(\tau_5) = 0.0086548$.

4. Conclusions

It was shown that also in the difficult case when $c_2 = 0$ it is possible to obtain solution of the problem. In the article [1] it was proved that the condition $c_2 = 0$ gives minimal dynamic error for the nth order equation $(n = 2, 3, \dots)$. For that reason this particular case is very important. The 3-rd order equation was analyzed and for the different kinds of roots the analytical formulae of the extremum of dynamic error $x(\tau)$ and time τ has been obtained. In the figures there are shown the transients of $x(t)$. The practical example, which was considered in [2], is also solved for the initial conditions $c_1 = 1$, $c_2 = 0$. The roots of the characteristic equation may be shifted in the desired location using the well known methods of the poles and zeros locations (see [7]).

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