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On dual approach to piecewise-linear elasto-plasticity. Part I: Continuum models

A. BORKOWSKI*

Institute of Fundamental Technological Problems, Polish Academy of Sciences, 21 Świętokrzyska St., 00–049 Warsaw, Poland

Abstract. This paper presents revised and extended version of theory proposed in the late 1970-ties by A. Čyras and his co-workers. This theory, based upon the notion of duality in mathematical programming, allows us to generate variational principles and to investigate existence and uniqueness of solutions for the broad class of problems of elasticity and plasticity. The paper covers analysis of solids made of linear elastic, elastic-strain hardening, elastic-perfectly plastic and rigid-perfectly plastic material. The novelty with respect to Čyras's theory lies in taking into account loads dispersed over the volume and displacements enforced on the part of surface. A new interpretation of optimum load for a rigid-perfectly plastic body is also given.

Keywords: imposed displacements, energy principles, mathematical programming.

1. Introduction

Already in late 1960-ties A. Čyras has shown close ties between the notion of duality in linear programming (LP) and the static-kinematic theorems for the ultimate load carrying capacity and optimum design of skeletal structures made of a rigid-perfectly plastic material [1]. Later this theory was extended by Čyras and his co-workers for linear elasticity and elastic-perfectly plastic behaviour. This required the replacement of the basic formalism by the more general theory of non-linear programming (NLP). The book published in 1974 by Čyras, Karkauskas and the present author comprised a complete unified model starting with dual variational principles for solids and ending with discrete descriptions of certain classes of structures like trusses, frames, plates and shells [2]. Recently an English translation of this book in memoriam of A. Čyras was published in Lithuania [3].

Modelling structural behaviour in terms of dual problems of mathematical programming drew attention of many authors in the period 1970–1990. Apart of Čyras and his group, the main contributions were made by G. Duvaut and J.-L. Lions [4] and B. Noble and M.J. Sewell [5], who investigated duality at the continuum level, as well as by G. Maier, who showed in his milestone paper [6] that all discrete models of structures governed by piecewise-linear constitutive laws can be interpreted as linear complementarity (LC) problems and converted into dual pairs of quadratic programming (QP) problems. A comprehensive state-of-art overview of that time can be found in [7]. The monograph [8] of the present author covered applications for skeletal structures. More recently many new results were obtained by P. Panagiotopoulos [9].

The aim of the present paper is to extend the approach proposed by Čyras taking into account a volume loading, e.g. the self-weight, that was neglected in the original theory. A new derivation of the load optimisation problem for rigid-perfectly plastic solids is also presented. The formulation of this problem in [2] violated the traction-displacement complementarity principle.

2. Notation and formulation

Let us consider a deformable solid depicted in Fig. 1. It occupies a volume V bounded by a surface S. We choose a fixed Cartesian reference frame $\mathbf{x} = (x_1, x_2, x_3)$ and assume that displacements $\mathbf{u} = (u_1, u_2, u_3)$ remain small. Then, strains $\boldsymbol{\varepsilon} = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23})$ are defined through a kinematic equation

$$\boldsymbol{\varepsilon} = \mathbf{C_V u} \tag{1}$$

Here

$$\mathbf{C_V} = \begin{bmatrix} \partial/\partial x_1 & 0 & 0 \\ 0 & \partial/\partial x_2 & 0 \\ 0 & 0 & \partial/\partial x_3 \\ \partial/\partial x_2 & \partial/\partial x_1 & 0 \\ \partial/\partial x_3 & 0 & \partial/\partial x_1 \\ 0 & \partial/\partial x_3 & \partial/\partial x_2 \end{bmatrix} \tag{2}$$

is a linear differential operator of kinematics.

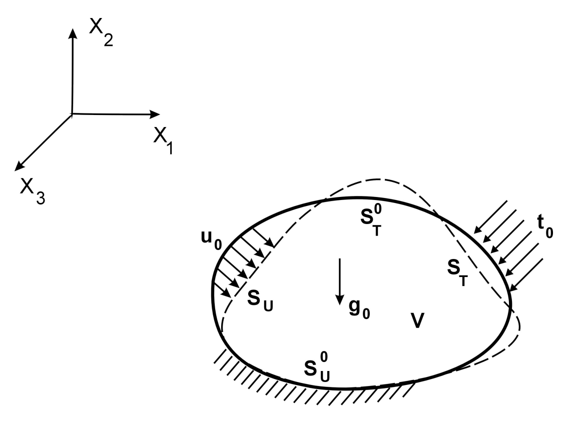

Fig. 1. Deformable body under static and kinematic load

* e-mail: abork@ippt.gov.pl

Let volume forces be denoted by $\mathbf{g} = (g_1, g_2, g_3)$. Then, stresses $\boldsymbol{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})$ must satisfy the equilibrium equation

$$\mathbf{g} = \mathbf{C}^*_\mathbf{V}\boldsymbol{\sigma} \tag{3}$$

inside the volume V. Here $\mathbf{C}^*_\mathbf{V}$ (see Eq. (4) at the bottom of this page) is a linear differential operator of volume equilibrium. The asterisk * indicates that this operator is adjoint to the kinematic operator $\mathbf{C}_\mathbf{V}$. The reason for replacing the strain and stress tensors by the single-column matrices $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ is that this allows us to use an uniform matrix notation throughout Part I and Part II of the present paper.

The equilibrium on the surface S is described by the equation

$$\mathbf{t} = \mathbf{C}^\mathbf{T}_\mathbf{S}\boldsymbol{\sigma} \tag{5}$$

where

$$\mathbf{C}^\mathbf{T}_\mathbf{S} = \begin{bmatrix} n_1 & 0 & 0 & n_2 & n_3 & 0 \\ 0 & n_2 & 0 & n_1 & 0 & n_3 \\ 0 & 0 & n_3 & 0 & n_1 & n_2 \end{bmatrix}, \tag{6}$$

is a linear algebraic operator (ordinary matrix), obtained by replacing derivatives in $\mathbf{C}^*_\mathbf{V}$ by the entries of unit vector $\mathbf{n} = (n_1, n_2, n_3)$ normal to S.

In the sequel we assume that the volume load $\mathbf{g_0}$ is always given (the subscript zero will indicate a given quantity). In particular, we may neglect this kind of loading taking $\mathbf{g_0} = \mathbf{0}$. We split the surface S of the body into two parts: S_T where the traction $\mathbf{t_0}$ is prescribed and S_U where the displacement $\mathbf{u_0}$ is given. In order to exclude a possibility of rigid body motion ($\boldsymbol{\varepsilon} = \mathbf{0}$ under $\mathbf{u} \neq \mathbf{0}$), prescribed displacements must vanish on a part of S_U.

By static load we understand $(\mathbf{g_0}, \mathbf{t_0})$ acting on the body, whereas by kinematic load we mean displacements $\mathbf{u_0}$ enforced on a part of its surface.

3. Linear-elastic solid

Just for the sake of reference let us recall the known facts from the theory of linear elasticity. The Hooke's law (Fig. 2.a) in our notation reads

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} \tag{7}$$

or

$$\boldsymbol{\varepsilon} = \mathbf{E}^{-1}\boldsymbol{\sigma} \tag{8}$$

where $\mathbf{E}$ is an elasticity matrix.

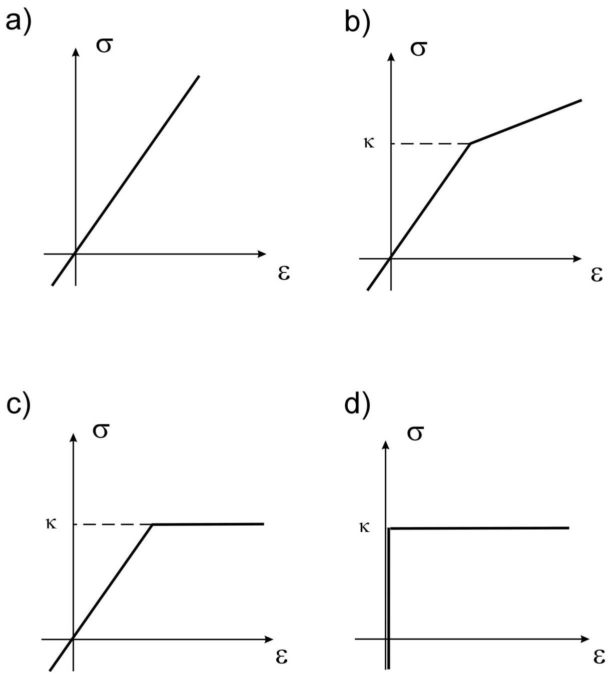

Fig. 2. Models of material: a) linear elastic, b) elastic-strain hardening, c) elastic-perfectly plastic, d) rigid-perfectly plastic

Table 1 summarises the derivation of dual energy principles for a linear-elastic solid. This derivation follows the template given in Table A3 of the Appendix. By eliminating strains from the complete set of governing equations, we obtain the reduced system governed by a self-adjoint operator. The potential L is chosen in such a way that its variation with respect to displacements generates the left-hand side of the first equation of the reduced system and its variation with respect to stresses yields the left-hand side of the second equation. The saddle point problem corresponds to the free variational principle for displacements, stresses and tractions. The primal problem expresses the principle of minimum potential energy and the dual problem (after the sign of the functional is changed) — the principle of minimum of a difference between the complementary energy and the work done by surface tractions on the prescribed displacements.

$$\mathbf{C}^*_\mathbf{V} = \begin{bmatrix} -\partial/\partial x_1 & 0 & 0 & -\partial/\partial x_2 & -\partial/\partial x_3 & 0 \\ 0 & -\partial/\partial x_2 & 0 & -\partial/\partial x_1 & 0 & -\partial/\partial x_3 \\ 0 & 0 & -\partial/\partial x_3 & 0 & -\partial/\partial x_1 & -\partial/\partial x_2 \end{bmatrix} \tag{4}$$

Table 1
Linear elastic solid under static and kinematic loads

Governing relations:

constitutive	$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon} = \mathbf{E}^{-1}\boldsymbol{\sigma}$	in V
kinematics	$\mathbf{C_V u} = \boldsymbol{\varepsilon}$	in V
equilibrium	$\mathbf{C^*_V}\boldsymbol{\sigma} = \mathbf{g}_0$	in V
	$\mathbf{C^T_S}\boldsymbol{\sigma} = \mathbf{t}$	on S
static boundary conditions	$\mathbf{t} = \mathbf{t}_0$	on S_T
kinematic boundary conditions	$\mathbf{u} = \mathbf{u}_0$	on S_U

Reduced system of relations in V:

	$\mathbf{u}$	$\boldsymbol{\sigma}$	1	
$\partial \mathbf{L}_u$		$\mathbf{C}^*_V$	$-\mathbf{g}_0$	$= \mathbf{0}$
$\partial \mathbf{L}_\sigma =$	$\mathbf{C}_V$	$-\mathbf{E}^{-1}$		$= \mathbf{0}$

Potential:

$$L(\mathbf{u},\boldsymbol{\sigma},\mathbf{t}) = -\frac{1}{2}\int_V \boldsymbol{\sigma}^T\mathbf{E}^{-1}\boldsymbol{\sigma}dV + \int_V \mathbf{u}^\mathbf{T}(\mathbf{C^*_V}\boldsymbol{\sigma} - \mathbf{g}_0)dV + \int_{S_T} \mathbf{u}^\mathbf{T}(\mathbf{C^T_S}\boldsymbol{\sigma} - \mathbf{t}_0)dS + \int_{S_U} \mathbf{u}^\mathbf{T}(\mathbf{C^T_S}\boldsymbol{\sigma} - \mathbf{t})dS + \int_{S_U} \mathbf{u}_0^\mathbf{T}\mathbf{t}dS$$

Saddle point:

$$L(\mathbf{u}_*,\boldsymbol{\sigma}_*,\mathbf{t}_*) = \inf_{\mathbf{u}}\sup_{\boldsymbol{\sigma},\mathbf{t}} L(\mathbf{u},\boldsymbol{\sigma},\mathbf{t})$$

Primal problem:

$$\text{find } \inf_{\mathbf{u},\boldsymbol{\sigma}}\left\{\frac{1}{2}\int_V \boldsymbol{\sigma}^\mathbf{T}\mathbf{E}^{-1}\boldsymbol{\sigma}dV - \int_V \mathbf{u}^\mathbf{T}\mathbf{g}_0 dV - \int_{S_T}\mathbf{u}^\mathbf{T}\mathbf{t}_0 dS\right\}$$

subject to

$\mathbf{C_V u} - \mathbf{E}^{-1}\boldsymbol{\sigma} = \mathbf{0}$ in V

$\mathbf{u} = \mathbf{u}_0$ on S_U

Dual problem:

$$\text{find } \sup_{\boldsymbol{\sigma},\mathbf{t}}\left\{-\frac{1}{2}\int_V \boldsymbol{\sigma}^T\mathbf{E}^{-1}\boldsymbol{\sigma}dV + \int_{S_U}\mathbf{t}^\mathbf{T}\mathbf{u}_0 dS\right\}$$

subject to

$\mathbf{C^*_V}\boldsymbol{\sigma} = \mathbf{g}_0$ in V

$\mathbf{C^T_S}\boldsymbol{\sigma} = \mathbf{t}_0$ on S_T

$\mathbf{C^T_S}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0}$ on S_U

The classical principles of minimum potential energy and minimum complementary energy are usually formulated in the presence of static load only. Taking $u_0 = 0$ in Table 1, we obtain them as the following dual principles:

$$\text{find } \inf_{\mathbf{u},\sigma}\left\{\frac{1}{2}\int_V \boldsymbol{\sigma}^\mathbf{T}\mathbf{E}^{-1}\boldsymbol{\sigma}dV - \int_V \mathbf{u}^\mathbf{T}\mathbf{g}_0 dV - \int_{S_T}\mathbf{u}^\mathbf{T}\mathbf{t}_0 dS\right\}$$

subject to

$\mathbf{C_V u} - \mathbf{E}^{-1}\boldsymbol{\sigma} = \mathbf{0}$ in V (9)

$\mathbf{u} = \mathbf{0}$ on S_U,

$$\text{find } \sup_{\boldsymbol{\sigma},\mathbf{t}}\left\{-\frac{1}{2}\int_V \boldsymbol{\sigma}^T\mathbf{E}^{-1}\boldsymbol{\sigma}dV\right\}$$

subject to

$\mathbf{C^*_V}\boldsymbol{\sigma} = \mathbf{g}_0$ in V (10)

$\mathbf{C^T_S}\boldsymbol{\sigma} = \mathbf{t}_0$ on S_T

$\mathbf{C^T_S}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0}$ on S_U.

Reactive tractions on the fixed part of the surface do not contribute to the functionals (they produce no work). Hence, the constraints on S_U can be omitted in (9), if we are not interested in obtaining reactions $\mathbf{t}_*$ from the static principle (10).

If we are interested in purely kinematic type of loading, then we have to assume $\mathbf{g_0} = \mathbf{0}$ and $\mathbf{t_0} = \mathbf{0}$ in Table 1. Then the dual principles read:

$$\text{find } \inf_{\mathbf{u},\sigma} \left\{ \frac{1}{2} \int_V \boldsymbol{\sigma}^\mathbf{T} \mathbf{E}^{-1} \boldsymbol{\sigma} dV \right\}$$

subject to (11)

$$\mathbf{C_V u} - \mathbf{E}^{-1}\boldsymbol{\sigma} = \mathbf{0} \text{ in } V$$
$$\mathbf{u} = \mathbf{u}_0 \text{ on } S_U.$$

$$\text{find } \sup_{\mathbf{t},\sigma} \left\{ -\frac{1}{2} \int_V \boldsymbol{\sigma}^T \mathbf{E}^{-1} \boldsymbol{\sigma} dV + \int_{S_U} \mathbf{t}^\mathbf{T} \mathbf{u}_0 dS \right\}$$

subject to (12)

$$\mathbf{C}^*_\mathbf{V}\boldsymbol{\sigma} = \mathbf{0} \text{ in } V$$
$$\mathbf{C}^\mathbf{T}_\mathbf{S}\boldsymbol{\sigma} = \mathbf{0} \text{ on } S_T$$
$$\mathbf{C}^\mathbf{T}_\mathbf{S}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0} \text{ on } S_U.$$

Note that modifying this model for unilaterally imposed displacements is formally simple — unknown tractions on S_U become non-negative:

$$\text{find } \inf_{\mathbf{u},\sigma} \left\{ \frac{1}{2} \int_V \boldsymbol{\sigma}^\mathbf{T} \mathbf{E}^{-1} \boldsymbol{\sigma} dV \right\}$$

subject to (13)

$$\mathbf{C_V u} - \mathbf{E}^{-1}\boldsymbol{\sigma} = \mathbf{0} \text{ in } V$$
$$\mathbf{u} \geqslant \mathbf{u}_0 \text{ on } S_U,$$

$$\text{find } \sup_{\mathbf{t},\sigma} \left\{ -\frac{1}{2} \int_V \boldsymbol{\sigma}^T \mathbf{E}^{-1} \boldsymbol{\sigma} dV + \int_{S_U} \mathbf{t}^\mathbf{T} \mathbf{u}_0 dS \right\}$$

subject to (14)

$$\mathbf{C}^*_\mathbf{V}\boldsymbol{\sigma} = \mathbf{0} \text{ in } V$$
$$\mathbf{C}^\mathbf{T}_\mathbf{S}\boldsymbol{\sigma} = \mathbf{0} \text{ on } S_T$$
$$\mathbf{C}^\mathbf{T}_\mathbf{S}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0}, \ \mathbf{t} \geqslant \mathbf{0} \text{ on } S_U.$$

This subtle modification has very serious consequences. Variational models given in Table 1 as well as their special cases (9)–(10) and (11)–(12) fall into the classical variational theory. If all constraints are of equality type and all variables are free in terms of their sign, then constrained problems can be replaced by unconstrained problems and their solutions can be found solving systems of linear equations. This does not apply for unilateral model (13)–(14): this model falls into the category of variational inequalities and its derivation is governed by the LCP-type scheme given in Table A4 of the Appendix. The appearance of inequality constraints and the presence of non-negative variables precludes its reduction to any set of equations.

4. Elastic-strain hardening solid

Let us consider a solid made of material that under uniaxial test expresses bi-linear behaviour, as shown in Fig. 2.b. This model can be seen as a simplified description of elastic-strain hardening material when the possibility of local unloading is neglected.

We assume a piecewise-linear plastic potential

$$\boldsymbol{\varphi} = \mathbf{H}\boldsymbol{\lambda} - \mathbf{N}^\mathbf{T}\boldsymbol{\sigma} + \boldsymbol{\kappa}_0 \tag{15}$$

where $\mathbf{H}$ is positive definite $(l \times l)$-matrix of strain hardening, vector $\boldsymbol{\lambda}$ contains l plastic multipliers, the columns of $(6 \times l)$-matrix $\mathbf{N}$ are gradients of the yield surface and vector $\boldsymbol{\kappa}_0$ contains l prescribed plastic modulae. The vanishing value of initial plastic potential

$$\boldsymbol{\varphi}_\mathbf{0} = -\mathbf{N}^\mathbf{T}\boldsymbol{\sigma} + \boldsymbol{\kappa}_0 = \mathbf{0} \tag{16}$$

defines a convex polyhedron that can be inscribed into a non-linear yield surface. Figure 3 shows an example of hexagon ($l = 6$) inscribed into the Huber-Mises yield surface. The advantage of this approximation is that the resulting models fall into the class of quadratic programming problems. Numerical solution of QP-problems is much easier that the solution of general non-linear programming problems.

Now the strain can be assumed as a sum of elastic and plastic components

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_\mathbf{E} + \boldsymbol{\varepsilon}_\mathbf{P} \tag{17}$$

where $\boldsymbol{\varepsilon}_\mathbf{E}$ is defined by Eq. (8) and

$$\boldsymbol{\varepsilon}_\mathbf{P} = \mathbf{N}\boldsymbol{\lambda} \tag{18}$$

with additional constraints

$$\boldsymbol{\varphi} \geqslant \mathbf{0}, \ \boldsymbol{\lambda} \geqslant \mathbf{0}, \ \boldsymbol{\varphi}^\mathbf{T}\lambda = 0 \tag{19}$$

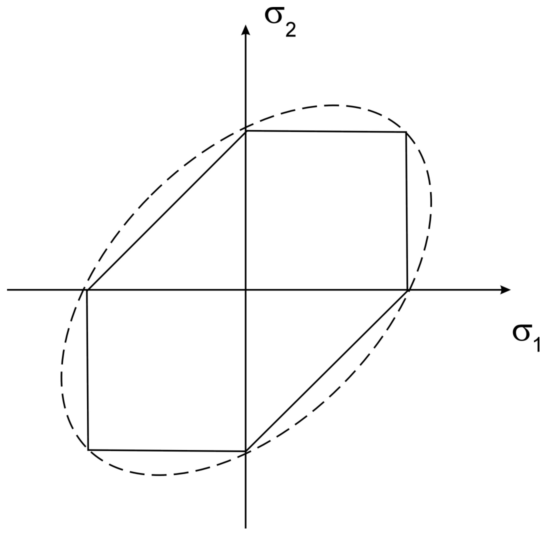

Fig. 3. Smooth and piecewise-linear yield curves

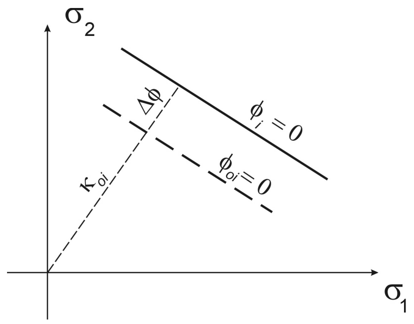

Fig. 4. Geometric interpretation of strain hardening

The adopted model of strain hardening is shown in Fig. 4. Plastic modulus κ_{0i} defines the distance of the i-th initial yield plane from the origin. Plastic strain shifts this plane to the outward of the yield surface by

$$\Delta\varphi_i = \sum_{j=1}^{l} H_{ij}\lambda_j \tag{20}$$

If the strain hardening matrix $\mathbf{H}$ is diagonal, then yield planes move independently. Otherwise, there is mutual interaction between them.

Due to the presence of inequality constraints and non-negative variables, the proper template for the derivation of variational principles is this time Table A4 of the Appendix. The result of this derivation is given in Table 2. Plastic multipliers and displacement play the role of x-variables of the LCP-problem. Stresses remain y-variables as they were in linear elasticity. The hardening matrix is positively definite, the inverse of elasticity matrix taken with minus sign is negative definite. Thus, all requirements of dual approach are met.

In the primal problem we have to minimise the difference between the sum of energy dissipated plastically and stored elastically and the work done by the static loading. The constraints of this problem ensure kinematic compatibility of strains and displacements. The functional of the dual problem expresses the difference between internally dissipated and stored energy and the work done by tractions on prescribed displacements. The constraints ensure static admissibility and equilibrium of stresses and surface tractions. No reduction to systems of equations is possible, since we deal with the LC-problem.

For purely static loading the dual model has the following shape:

$$\begin{aligned}
&\text{find} \inf_{\lambda,\mathbf{u},\sigma}\left\{\frac{1}{2}\int_V \boldsymbol{\lambda}^{\mathbf{T}}\mathbf{H}\boldsymbol{\lambda}dV + \frac{1}{2}\int_V \boldsymbol{\sigma}^{\mathbf{T}}\mathbf{E}^{-1}\boldsymbol{\sigma}dV \right.\\
&\left. + \int_V \boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{\kappa}_0 dV - \int_V \mathbf{u}^{\mathbf{T}}\mathbf{g}_0 dV - \int_{S_T} \mathbf{u}^{\mathbf{T}}\mathbf{t}_0 dS\right\}\\
&\text{subject to}\\
&\mathbf{C_V u} - \mathbf{E}^{-1}\boldsymbol{\sigma} - \mathbf{N}\boldsymbol{\lambda} = \mathbf{0} \text{ in } V\\
&\boldsymbol{\lambda} \geqslant \mathbf{0} \text{ in } V\\
&\mathbf{u} = \mathbf{0} \text{ on } S_U,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\text{find} \sup_{\lambda,\sigma,\mathbf{t}}\left\{-\frac{1}{2}\int_V \boldsymbol{\lambda}^{\mathbf{T}}\mathbf{H}\boldsymbol{\lambda}dV - \frac{1}{2}\int_V \boldsymbol{\sigma}^{T}\mathbf{E}^{-1}\boldsymbol{\sigma}dV\right\}\\
&\text{subject to}\\
&-\mathbf{H}\boldsymbol{\lambda} + \mathbf{N}^{\mathbf{T}}\boldsymbol{\sigma} \leqslant \boldsymbol{\kappa}_0 \text{ in } V\\
&\mathbf{C}^*_{\mathbf{V}}\boldsymbol{\sigma} = \mathbf{g}_0 \text{ in } V\\
&\mathbf{C}^{\mathbf{T}}_{\mathbf{S}}\boldsymbol{\sigma} = \mathbf{t}_0 \text{ on } S_T\\
&\mathbf{C}^{\mathbf{T}}_{\mathbf{S}}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0} \text{ on } S_U.
\end{aligned} \tag{22}$$

Again, the constraints on S_U can be omitted, when we are not interested in reactions.

When the loading is purely kinematic, the dual model reduces to:

$$\begin{aligned}
&\text{find} \inf_{\lambda,\mathbf{u},\sigma}\left\{\frac{1}{2}\int_V \boldsymbol{\lambda}^{\mathbf{T}}\mathbf{H}\boldsymbol{\lambda}dV + \frac{1}{2}\int_V \boldsymbol{\sigma}^{\mathbf{T}}\mathbf{E}^{-1}\boldsymbol{\sigma}dV \right.\\
&\left. + \int_V \boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{\kappa}_0 dV\right\}\\
&\text{subject to}\\
&\mathbf{C_V u} - \mathbf{E}^{-1}\boldsymbol{\sigma} - \mathbf{N}\boldsymbol{\lambda} = \mathbf{0} \text{ in } V\\
&\boldsymbol{\lambda} \geqslant \mathbf{0} \text{ in } V\\
&\mathbf{u} = \mathbf{u}_0 \text{ on } S_U,
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\text{find} \sup_{\lambda,\sigma,\mathbf{t}}\left\{-\frac{1}{2}\int_V \boldsymbol{\lambda}^{\mathbf{T}}\mathbf{H}\boldsymbol{\lambda}dV - \frac{1}{2}\int_V \boldsymbol{\sigma}^{T}\mathbf{E}^{-1}\boldsymbol{\sigma}dV \right.\\
&\left. + \int_{S_U} \mathbf{t}^{\mathbf{T}}\mathbf{u}_0 dS\right\}\\
&\text{subject to}\\
&-\mathbf{H}\boldsymbol{\lambda} + \mathbf{N}^{\mathbf{T}}\boldsymbol{\sigma} \leqslant \boldsymbol{\kappa}_0 \text{ in } V\\
&\mathbf{C}^*_{\mathbf{V}}\boldsymbol{\sigma} = \mathbf{0} \text{ in } V\\
&\mathbf{C}^{\mathbf{T}}_{\mathbf{S}}\boldsymbol{\sigma} = \mathbf{0} \text{ on } S_T\\
&\mathbf{C}^{\mathbf{T}}_{\mathbf{S}}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0} \text{ on } S_U.
\end{aligned} \tag{24}$$

This time the energy dissipated and stored in the body is minimised in the primal variational principle. The solution of the dual principle is to be found in the class of self-equilibrated stresses. The solution $(\boldsymbol{\lambda}_*, \mathbf{u}_*, \boldsymbol{\sigma}_*, \mathbf{t}_*)$ of the dual problems (23)–(24) exists for any $\mathbf{u_0}$. Due to convexity-concavity of the cost functionals this solution is unique.

Table 2
Elastic-strain hardening solid under static and kinematic load

Governing relations:

constitutive	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_P$	in V
	$\boldsymbol{\varepsilon}_E = \mathbf{E}^{-1}\boldsymbol{\sigma},\ \boldsymbol{\varepsilon}_P = \mathbf{N}\boldsymbol{\lambda}$	in V
	$\boldsymbol{\varphi} = \mathbf{H}\boldsymbol{\lambda} - \mathbf{N}^{\mathbf{T}}\boldsymbol{\sigma} + \boldsymbol{\kappa}_0$	in V
	$\boldsymbol{\varphi} \geqslant \mathbf{0},\ \boldsymbol{\lambda} \geqslant \mathbf{0},\ \boldsymbol{\varphi}^{\mathbf{T}}\boldsymbol{\lambda} = 0$	in V
kinematics	$\mathbf{C}_{\mathbf{V}}\mathbf{u} = \boldsymbol{\varepsilon}$	in V
equilibrium	$\mathbf{C}_{\mathbf{V}}^{\mathbf{T}}\boldsymbol{\sigma} = \mathbf{g}_0$	in V
	$\mathbf{C}_{\mathbf{S}}^{\mathbf{T}}\boldsymbol{\sigma} = \mathbf{t}$	on S
static boundary conditions	$\mathbf{t} = \mathbf{t}_0$	on S_T
kinematic boundary conditions	$\mathbf{u} = \mathbf{u}_0$	on S_U

Reduced system of relations in V:

	$\boldsymbol{\lambda} \geqslant \mathbf{0}$	$\mathbf{u}$	$\boldsymbol{\sigma}$	1	
$\partial \mathbf{L}_\lambda =$	$\mathbf{H}$		$-\mathbf{N}^{\mathbf{T}}$	$\boldsymbol{\kappa}_0$	$\geqslant \mathbf{0}$
$\partial \mathbf{L}_u =$			$\mathbf{C}_V^*$	$-\mathbf{g}_0$	$= \mathbf{0}$
$\partial \mathbf{L}_\sigma =$	$-\mathbf{N}$	$\mathbf{C}_V$	$-\mathbf{E}^{-1}$		$= \mathbf{0}$

Potential:

$$L(\boldsymbol{\lambda}, \mathbf{u}, \boldsymbol{\sigma}, \mathbf{t}) = \frac{1}{2}\int_V \boldsymbol{\lambda}^{\mathbf{T}}\mathbf{H}\boldsymbol{\lambda}\,dV - \frac{1}{2}\int_V \boldsymbol{\sigma}^T\mathbf{E}^{-1}\boldsymbol{\sigma}\,dV - \int_V \boldsymbol{\lambda}\mathbf{N}^{\mathbf{T}}\boldsymbol{\sigma}\,dV + \int_V \boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{\kappa}_0\,dV + \int_V \mathbf{u}^{\mathbf{T}}(\mathbf{C}_V^*\boldsymbol{\sigma} - \mathbf{g}_0)\,dV$$
$$+ \int_{S_T} \mathbf{u}^{\mathbf{T}}(\mathbf{C}_{\mathbf{S}}^{\mathbf{T}}\boldsymbol{\sigma} - \mathbf{t}_0)\,dS + \int_{S_U} \mathbf{u}^{\mathbf{T}}(\mathbf{C}_{\mathbf{S}}^{\mathbf{T}}\boldsymbol{\sigma} - \mathbf{t})\,dS + \int_{S_U} \mathbf{u}_0^{\mathbf{T}}\mathbf{t}\,dS$$

Saddle point:

$$L(\boldsymbol{\lambda}_*, \mathbf{u}_*, \boldsymbol{\sigma}_*, \mathbf{t}_*) = \inf_{\boldsymbol{\lambda}\geqslant\mathbf{0},\mathbf{u}} \ \sup_{\boldsymbol{\sigma},\mathbf{t}} L(\boldsymbol{\lambda}, \mathbf{u}, \boldsymbol{\sigma}, \mathbf{t})$$

Primal problem:

$$\text{find} \inf_{\boldsymbol{\lambda},\mathbf{u},\boldsymbol{\sigma}} \left\{ \frac{1}{2}\int_V \boldsymbol{\lambda}^{\mathbf{T}}\mathbf{H}^{-\mathbf{1}}\boldsymbol{\lambda}\,dV + \frac{1}{2}\int_V \boldsymbol{\sigma}^T\mathbf{E}^{-\mathbf{1}}\boldsymbol{\sigma}\,dV + \int_V \boldsymbol{\lambda}^T\boldsymbol{\kappa}_0\,dV - \int_V \mathbf{u}^{\mathbf{T}}\mathbf{g}_0\,dV - \int_{S_T} \mathbf{u}^{\mathbf{T}}\mathbf{t}_0\,dS \right\}$$

subject to

$\mathbf{C}_{\mathbf{V}}\mathbf{u} - \mathbf{E}^{-\mathbf{1}}\boldsymbol{\sigma} - \mathbf{N}\boldsymbol{\lambda} = \mathbf{0}$ in V

$\boldsymbol{\lambda} \geqslant \mathbf{0}$ in V

$\mathbf{u} = \mathbf{u}_0$ on S_U

Dual problem:

$$\text{find} \sup_{\boldsymbol{\lambda},\boldsymbol{\sigma},\mathbf{t}} \left\{ -\frac{1}{2}\int_V \boldsymbol{\lambda}^T\mathbf{H}^{-\mathbf{1}}\boldsymbol{\lambda}\,dV - \frac{1}{2}\int_V \boldsymbol{\sigma}^T\mathbf{E}^{-\mathbf{1}}\boldsymbol{\sigma}\,dV + \int_{S_U} \mathbf{t}^{\mathbf{T}}\mathbf{u}_0\,dS \right\}$$

subject to

$-\mathbf{H}\boldsymbol{\lambda} + \mathbf{N}^T\boldsymbol{\sigma} \geqslant \boldsymbol{\kappa}_0$ in V

$\mathbf{C}_{\mathbf{V}}^*\boldsymbol{\sigma} = \mathbf{g}_0$ in V

$\mathbf{C}_{\mathbf{S}}^{\mathbf{T}}\boldsymbol{\sigma} = \mathbf{t}_0$ on S_T

$\mathbf{C}_{\mathbf{S}}^{\mathbf{T}}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0}$ on S_U

5. Elastic-perfectly plastic solid

Taking $\mathbf{H} = \mathbf{0}$, we obtain perfectly plastic behaviour. Figure 2c shows such behaviour for uniaxial stress-strain state. Now plastic potential keeps its initial value (16) irrespective of plastic strains.

The stress admissibility conditions do not depend any more on plastic multipliers and, hence, on plastic strains. The sole matrix that appears on the diagonal of the reduced system of governing relations (Table 3) is the inverse matrix of elasticity. The potential of this system is strictly concave with respect to stresses due to negative quadratic term but it is only linear with respect to plastic multipliers. Hence, we can expect the uniqueness of the solution with respect to stresses only.

Moreover, we have lost the nice property of the strain-hardening solid that it is able to carry any loading. In the case of perfectly plastic material the yield surface remains fixed and for excessively high load there might be no statically admissible stress field. This means that the constraints of the dual problem become contradictory and the saddle point is not attainable.

For purely static loading the dual variational principles given in Table 3 take the following form:

$$\text{find } \inf_{\boldsymbol{\lambda},\mathbf{u},\boldsymbol{\sigma}} \left\{ \frac{1}{2}\int\limits_V \boldsymbol{\sigma}^{\mathbf{T}}\mathbf{E}^{-\mathbf{1}}\boldsymbol{\sigma}dV + \int\limits_V \boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{\kappa}_0 dV - \int\limits_V \mathbf{u}^T\mathbf{g}_0 dV - \int\limits_S \mathbf{u}^T\mathbf{t}_0 dS \right\} \tag{24}$$

subject to

$$\mathbf{C_V u} - \mathbf{E}^{-\mathbf{1}}\boldsymbol{\sigma} - \mathbf{N}\boldsymbol{\lambda} = \mathbf{0} \text{ in } V$$
$$\boldsymbol{\lambda} \geqslant \mathbf{0} \text{ in } V$$
$$\mathbf{u} = \mathbf{0} \text{ on } S_U$$

$$\text{find } \sup_{\boldsymbol{\sigma},\mathbf{t}} \left\{ -\frac{1}{2}\int\limits_V \boldsymbol{\sigma}^T\mathbf{E}^{-\mathbf{1}}\boldsymbol{\sigma}dV \right\}$$

subject to

$$\mathbf{N^T}\boldsymbol{\sigma} \leqslant \boldsymbol{\kappa}_0 \text{ in } V \tag{25}$$
$$\mathbf{C_V}^*\boldsymbol{\sigma} = \mathbf{g}_0 \text{ in } V$$
$$\mathbf{C_S^T}\boldsymbol{\sigma} = \mathbf{t}_0 \text{ on } S_T$$
$$\mathbf{C_S^T}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0} \text{ on } S_U$$

and for purely kinematic loading they read as follows:

$$\text{find } \inf_{\lambda,\mathbf{u},\sigma} \left\{ \frac{1}{2}\int\limits_V \boldsymbol{\sigma}^{\mathbf{T}}\mathbf{E}^{-\mathbf{1}}\boldsymbol{\sigma}dV + \int\limits_V \lambda^{\mathbf{T}}\boldsymbol{\kappa}_0 dV \right\}$$

subject to (26)

$$\mathbf{C_V u} - \mathbf{E}^{-\mathbf{1}}\boldsymbol{\sigma} - \mathbf{N}\boldsymbol{\lambda} = \mathbf{0} \text{ in } V$$
$$\lambda \geqslant \mathbf{0} \text{ in } V$$
$$\mathbf{u} = \mathbf{u}_0 \text{ on } S_U,$$

$$\text{find } \sup_{\boldsymbol{\sigma},\mathbf{t}} \left\{ -\frac{1}{2}\int\limits_V \boldsymbol{\sigma}^T\mathbf{E}^{-\mathbf{1}}\boldsymbol{\sigma}dV + \int\limits_{S_U} \mathbf{t^T u}_0 dS \right\}$$

subject to

$$\mathbf{N^T}\boldsymbol{\sigma} \leqslant \boldsymbol{\kappa}_0 \text{ in } V \tag{27}$$
$$\mathbf{C_V}^*\boldsymbol{\sigma} = \mathbf{0} \text{ in } V$$
$$\mathbf{C_S^T}\boldsymbol{\sigma} = \mathbf{0} \text{ on } S_T$$
$$\mathbf{C_S^T}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0} \text{ on } S_U.$$

Note that the energy dissipated plastically is now linear with respect to plastic multipliers (the second term in the functional of the primal problem). Again there is no warranty of the existence of solution, since self-equilibrated stresses might not be able to remain inside the fixed yield surface (the first constraint of the dual problem).

6. Rigid-perfectly plastic solid

Neglecting completely elastic strains, i.e. taking $\mathbf{E^{-1}} = \mathbf{0}$, we obtain rigid-perfectly plastic model (Fig. 2.d). It is better now to consider the rates of kinematic variables $\dot{\boldsymbol{\lambda}}$, $\dot{\mathbf{u}}$ instead of their values $\boldsymbol{\lambda}$, $\mathbf{u}$, since the plastic flow becomes unlimited after the stress state reaches the yield surface.

Table 4 contains the derivation of dual variational principles for a rigid-perfectly plastic solid subjected simultaneously to static and kinematic loading. Looking at the final result of this derivation, we notice several interesting features. First, the functionals of both dual problems are linear. Second, the duality reflects kinematic and static aspects of the problem: the primal variational principle contains only kinematic variables and constraints, whereas the dual one contains solely static variables and constraints. Hence, we can call them a kinematic variational principle and a static variational principle. Finally, the problem of finding response of such solid to a purely static load would be ill-posed. This is seen clearly from the functional of the static principle: this functional contains no term associated with $\mathbf{g_0}$ or $\mathbf{t_0}$. Hence, it would vanish for $\dot{\mathbf{u}}_\mathbf{0} = \mathbf{0}$.

Such result confirms what we know from the theory of ultimate load carrying capacity. If an arbitrary assigned static load $\mathbf{g_0}$, $\mathbf{t_0}$ falls below the ultimate load then the body remains rigid ($\dot{\boldsymbol{\lambda}}_* = \mathbf{0}$, $\dot{\mathbf{u}}_* = \mathbf{0}$) and stresses can not be determined uniquely. If $\mathbf{g_0}$, $\mathbf{t_0}$ exceeds the ultimate load then the constraints of the static principle become contradictory and no solution exists.

Table 3
Elastic-perfectly plastic solid under static and kinematic load

Governing relations:

constitutive	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_P$	in V
	$\boldsymbol{\varepsilon}_E = \mathbf{E}^{-1}\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}_P = \mathbf{N}\boldsymbol{\lambda}$	in V
	$\boldsymbol{\varphi} = -\mathbf{N}^{\mathbf{T}}\boldsymbol{\sigma} + \boldsymbol{\kappa}_0$	in V
	$\boldsymbol{\varphi} \geqslant \mathbf{0}$, $\boldsymbol{\lambda} \geqslant \mathbf{0}$, $\boldsymbol{\varphi}^{\mathbf{T}}\boldsymbol{\lambda} = 0$	in V
kinematics	$\mathbf{C_V u} = \boldsymbol{\varepsilon}$	in V
equilibrium	$\mathbf{C_V^T}\boldsymbol{\sigma} = \mathbf{g}_0$	in V
	$\mathbf{C_S^T}\boldsymbol{\sigma} = \mathbf{t}$	on S
static boundary conditions	$\mathbf{t} = \mathbf{t}_0$	on S_T
kinematic boundary conditions	$\mathbf{u} = \mathbf{u}_0$	on S_U

Reduced system of relations in V:

	$\boldsymbol{\lambda} \geqslant \mathbf{0}$	$\mathbf{u}$	$\boldsymbol{\sigma}$	1	
$\nabla\mathbf{L}_\lambda =$			$-\mathbf{N}^{\mathbf{T}}$	$\boldsymbol{\kappa}_0$	$\geqslant \mathbf{0}$
$\nabla\mathbf{L}_u =$			$\mathbf{C}_V^*$	$-\mathbf{g}_0$	$= \mathbf{0}$
$\nabla\mathbf{L}_\sigma =$	$-\mathbf{N}$	$\mathbf{C}_V$	$-\mathbf{E}^{-1}$		$= \mathbf{0}$

Potential:

$$L(\boldsymbol{\lambda}, \mathbf{u}, \boldsymbol{\sigma}, \mathbf{t}) = -\frac{1}{2}\int_V \boldsymbol{\sigma}^T\mathbf{E}^{-1}\boldsymbol{\sigma}dV - \int_V \boldsymbol{\lambda}\mathbf{N}^{\mathbf{T}}\boldsymbol{\sigma}dV + \int_V \boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{\kappa}_0 dV + \int_V \mathbf{u}^{\mathbf{T}}(\mathbf{C_V^*}\boldsymbol{\sigma} - \mathbf{g}_0)dV + \int_{S_T} \mathbf{u}^{\mathbf{T}}(\mathbf{C_S^T}\boldsymbol{\sigma} - \mathbf{t}_0)dS$$
$$+ \int_{S_U} \mathbf{u}^{\mathbf{T}}(\mathbf{C_S^T}\boldsymbol{\sigma} - \mathbf{t})dS + \int_{S_U} \mathbf{u}_0^{\mathbf{T}}\mathbf{t}dS$$

Saddle point:

$$L(\boldsymbol{\lambda}_*, \mathbf{u}_*, \boldsymbol{\sigma}_*, \mathbf{t}_*) = \inf_{\boldsymbol{\lambda}\geqslant\mathbf{0},\mathbf{u}} \sup_{\boldsymbol{\sigma},\mathbf{t}} L(\boldsymbol{\lambda}, \mathbf{u}, \boldsymbol{\sigma}, \mathbf{t})$$

Primal problem:

$$\text{find } \inf_{\boldsymbol{\lambda},\mathbf{u},\boldsymbol{\sigma}} \left\{ \frac{1}{2}\int_V \boldsymbol{\sigma}^{\mathbf{T}}\mathbf{E}^{-1}\boldsymbol{\sigma}dV + \int_V \boldsymbol{\lambda}^T\boldsymbol{\kappa}_0 dV - \int_V \mathbf{u}^{\mathbf{T}}\mathbf{g}_0 dV - \int_{S_T} \mathbf{u}^{\mathbf{T}}\mathbf{t}_0 dS \right\}$$

subject to

$\mathbf{C_V u} - \mathbf{E}^{-1}\boldsymbol{\sigma} - \mathbf{N}\boldsymbol{\lambda} = \mathbf{0}$ in V

$\boldsymbol{\lambda} \geqslant \mathbf{0}$ in V

$\mathbf{u} = \mathbf{u}_0$ on S_U

Dual problem:

$$\text{find } \sup_{\boldsymbol{\lambda},\boldsymbol{\sigma},\mathbf{t}} \left\{ -\frac{1}{2}\int_V \boldsymbol{\sigma}^T\mathbf{E}^{-1}\boldsymbol{\sigma}dV + \int_{S_U} \mathbf{t}^{\mathbf{T}}\mathbf{u}_0 dS \right\}$$

subject to

$\mathbf{N}^T\boldsymbol{\sigma} \leqslant \boldsymbol{\kappa}_0$ in V

$\mathbf{C_V^*}\boldsymbol{\sigma} = \mathbf{g}_0$ in V

$\mathbf{C_S^T}\boldsymbol{\sigma} = \mathbf{t}_0$ on S_T

$\mathbf{C_S^T}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0}$ on S_U

Table 4
Rigid-perfectly plastic solid under static and kinematic load

Governing relations:

constitutive	$\dot{\varepsilon}_{\mathbf{P}} = \mathbf{N}\dot{\boldsymbol{\lambda}}$	in V
	$\boldsymbol{\varphi} = -\mathbf{N}^{\mathbf{T}}\boldsymbol{\sigma} + \boldsymbol{\kappa}_0$	in V
	$\boldsymbol{\varphi} \geqslant \mathbf{0},\ \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0},\ \boldsymbol{\varphi}^{\mathbf{T}}\dot{\boldsymbol{\lambda}} = 0$	in V
kinematics	$\mathbf{C_V}\dot{\mathbf{u}} = \dot{\varepsilon}$	in V
equilibrium	$\mathbf{C_V^T}\boldsymbol{\sigma} = \mathbf{g}_0$	in V
	$\mathbf{C_S^T}\boldsymbol{\sigma} = \mathbf{t}$	on S
static boundary conditions	$\mathbf{t} = \mathbf{t}_0$	on S_T
kinematic boundary conditions	$\dot{\mathbf{u}} = \dot{\mathbf{u}}_0$	on S_U

Reduced system of relations in V:

	$\dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}$	$\dot{\mathbf{u}}$	$\boldsymbol{\sigma}$	1	
$\nabla\mathbf{L}_{\dot{\lambda}} =$			$-\mathbf{N}^{\mathbf{T}}$	$\boldsymbol{\kappa}_0$	$\geqslant \mathbf{0}$
$\nabla\mathbf{L}_{\dot{u}} =$			$\mathbf{C}_V^*$	$-\mathbf{g}_0$	$= \mathbf{0}$
$\nabla\mathbf{L}_{\sigma} =$	$-\mathbf{N}$	$\mathbf{C}_V$			$= \mathbf{0}$

Potential:

$$L(\dot{\boldsymbol{\lambda}}, \dot{\mathbf{u}}, \boldsymbol{\sigma}, \mathbf{t}) = -\int_V \dot{\boldsymbol{\lambda}}\mathbf{N}^{\mathbf{T}}\boldsymbol{\sigma}dV + \int_V \dot{\boldsymbol{\lambda}}^{\mathbf{T}}\boldsymbol{\kappa}_0 dV + \int_V \dot{\mathbf{u}}^{\mathbf{T}}(\mathbf{C_V^*}\boldsymbol{\sigma} - \mathbf{g}_0)dV + \int_{S_T} \dot{\mathbf{u}}^{\mathbf{T}}(\mathbf{C_S^T}\boldsymbol{\sigma} - \mathbf{t}_0)dS + \int_{S_U} \dot{\mathbf{u}}^{\mathbf{T}}(C_S^T\boldsymbol{\sigma} - \mathbf{t})dS + \int_{S_U} \dot{\mathbf{u}}_0^{\mathbf{T}}\mathbf{t}dS$$

Saddle point:

$$L(\dot{\boldsymbol{\lambda}}_*, \dot{\mathbf{u}}_*, \boldsymbol{\sigma}_*, \mathbf{t}_*) = \inf_{\dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \dot{\mathbf{u}}} \sup_{\boldsymbol{\sigma}, \mathbf{t}} L(\dot{\boldsymbol{\lambda}}, \dot{\mathbf{u}}, \boldsymbol{\sigma}, \mathbf{t})$$

Primal problem:

$$\text{find} \inf_{\dot{\boldsymbol{\lambda}} \geqslant 0, \dot{\mathbf{u}}} \left\{ \int_V \dot{\boldsymbol{\lambda}}^T \boldsymbol{\kappa}_0 dV - \int_V \dot{\mathbf{u}}^{\mathbf{T}}\mathbf{g}_0 dV - \int_{S_T} \dot{\mathbf{u}}^{\mathbf{T}}\mathbf{t}_0 dS \right\}$$

subject to

$\mathbf{C_V}\dot{\mathbf{u}} - \mathbf{N}\dot{\boldsymbol{\lambda}} = \mathbf{0}$ in V

$\dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}$ in V

$\dot{\mathbf{u}} = \dot{\mathbf{u}}_0$ on S_U

Dual problem:

$$\text{find} \sup_{\boldsymbol{\sigma}, \mathbf{t}} \int_V \mathbf{t}^T \dot{\mathbf{u}}_0 dS$$

subject to

$\mathbf{N}^T\boldsymbol{\sigma} \leqslant \boldsymbol{\kappa}_0$ in V

$\mathbf{C_V^*}\boldsymbol{\sigma} = \mathbf{g}_0$ in V

$\mathbf{C_S^T}\boldsymbol{\sigma} = \mathbf{t}_0$ on S_T

$\mathbf{C_S^T}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0}$ on S_U

7. Optimum ultimate load

Assuming the loading to be purely kinematical, we obtain the following dual variational principles for the rigid-perfectly plastic solid:

$$\text{find } \inf_{\dot\lambda,\dot{\mathbf{u}}} \int_V \dot\lambda^{\mathbf{T}} \boldsymbol{\kappa}_0 dV$$

subject to

$$\begin{aligned} &\mathbf{C_V}\dot{\mathbf{u}} - N\dot{\boldsymbol{\lambda}} = \mathbf{0} \text{ in } V \\ &\dot{\boldsymbol{\lambda}} \geqslant \mathbf{0} \text{ in } V \\ &\dot{\mathbf{u}} = \dot{\mathbf{u}}_0 \text{ on } S_U, \end{aligned} \tag{28}$$

$$\text{find } \sup_{\boldsymbol{\sigma},\mathbf{t}} \int_{S_U} \mathbf{t}^{\mathbf{T}} \dot{\mathbf{u}}_0 dS$$

subject to

$$\begin{aligned} &\mathbf{N^T}\boldsymbol{\sigma} \leqslant \boldsymbol{\kappa}_0 \text{ in } V \\ &\mathbf{C_V}^*\boldsymbol{\sigma} = \mathbf{0} \text{ in } V \\ &\mathbf{C_S^T}\boldsymbol{\sigma} = \mathbf{0} \text{ on } S_T \\ &\mathbf{C_S^T}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0} \text{ on } S_U. \end{aligned} \tag{29}$$

These problems of constrained optimisation are variational counterparts of the linear programming problems: the extreme points of linear functionals are searched for over convex domains described by linear constraints.

Interestingly enough, model (28)–(29) can be interpreted in terms of optimum ultimate loading. The functional maximised in the static variational principle (29) can be interpreted twofold:

1. As an expression of the power (work rate) generated by surface tractions on prescribed displacement rates.
2. As an indicator of "quality" of surface tractions that bring the body to the state of plastic collapse.

In the latter case, $\dot{\mathbf{u}}_\mathbf{0}$ is merely a vector field of given cost coefficients and the problem is formulated in the following way: "given $\boldsymbol{\kappa}_\mathbf{0}$ in V and $\dot{\mathbf{u}}_\mathbf{0}$ on S_U find such distribution of surface tractions $\mathbf{t}_*$ on S_U that brings the body to plastic collapse and corresponds to $\sup \int_{S_U} \mathbf{t}^{\mathbf{T}}\dot{\mathbf{u}}_0 dS$. Since the cost functional can be seen as a weighted resultant of $\mathbf{t}$, we can say that in the model (28)–(29) we are looking for the most favourable distribution of the surface loading, i.e. for the distribution under which the body is able to carry maximum resultant load. In particular, we may assume a unit field of cost vectors, i.e. $\|\dot{\mathbf{u}}_\mathbf{0}\| = 1$ throughout S_U.

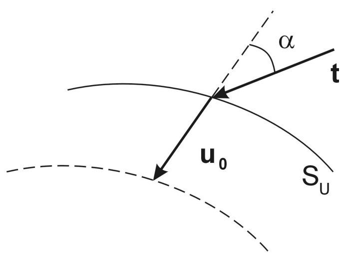

Fig. 5. Prescribed displacement and reaction on the surface of the body

An interesting question is whether freely optimised load will be co-linear with the prescribed displacement rate. Since the value of scalar product

$$\mathbf{t^T}\dot{\mathbf{u}}_\mathbf{0} = \|\mathbf{t}\| \cdot \|\dot{\mathbf{u}}_\mathbf{0}\| \cdot \cos\alpha \tag{30}$$

is largest for $\alpha = 0$ (compare Fig. 5), one should expect such result, at least for an isotropic material. However, plastic anisotropy may introduce a preferable direction that differs on S_U from the prescribed one. Then the direction of $\mathbf{t}_*$ need not coincide locally with the direction of $\dot{\mathbf{u}}_\mathbf{0}$.

Model (28)–(29) shows us that if we treat tractions as free variables then we must completely define displacement rates on the surface. If the direction of loading is given at each point of S_T, i.e. if

$$\mathbf{t} = t\tilde{\mathbf{t}}_\mathbf{0} \tag{31}$$

where t is an unknown load modulus and $\tilde{\mathbf{t}}_\mathbf{0}$ is a given unit vector of load direction, then the power of loading

$$\dot W = \int_{S_T} \dot{\mathbf{u}}^T \mathbf{t} dS = \int_{S_T} (\tilde{\mathbf{t}}_\mathbf{0}^\mathbf{T}\dot{\mathbf{u}}) t dS = \int_{S_T} v_0 t dS \tag{32}$$

becomes linear in t, provided the value of scalar product

$$v_0 = \tilde{\mathbf{t}}_\mathbf{0}^\mathbf{T}\dot{\mathbf{u}} \tag{33}$$

remains given at each point of S_T. Therefore, under prescribed direction of loading, the load-oriented problem is governed by the following pair of dual variational principles:

$$\text{find } \inf_{\dot\lambda,\dot{\mathbf{u}}} \int_V \dot{\boldsymbol{\lambda}}^{\mathbf{T}} \boldsymbol{\kappa}_0 dV$$

subject to

$$\begin{aligned} &\mathbf{C_V}\dot{\mathbf{u}} - \mathbf{N}\dot{\boldsymbol{\lambda}} = \mathbf{0} \text{ in } V \\ &\dot{\boldsymbol{\lambda}} \geqslant \mathbf{0} \text{ in } V \\ &\tilde{\mathbf{t}}_\mathbf{0}^\mathbf{T}\dot{\mathbf{u}} = \nu_0 \text{ on } S_T \\ &\dot{\mathbf{u}} = \mathbf{0} \text{ on } S_U \end{aligned} \tag{34}$$

$$\text{find } \sup_{\boldsymbol{\sigma},t,\mathbf{t}} \int_{S_U} \nu_0 t dS$$

subject to

$$\begin{aligned} &\mathbf{N^T}\boldsymbol{\sigma} \leqslant \boldsymbol{\kappa}_0 \text{ in } V \\ &\mathbf{C_V}^*\boldsymbol{\sigma} = \mathbf{0} \text{ in } V \\ &\mathbf{C_S^T}\boldsymbol{\sigma} = t\tilde{\mathbf{t}}_\mathbf{0} \text{ on } S_T \\ &\mathbf{C_S^T}\boldsymbol{\sigma} - \mathbf{t} = \mathbf{0} \text{ on } S_U \end{aligned} \tag{35}$$

Thus, when the direction of loading is given, then it is sufficient to prescribe ν_0 at each point of S_T. The simplest choice is to take $\nu_0 = 1$. This means that we are looking for the ultimate load with maximum resultant $\int_{S_T} t dS$.

Finally, let us introduce a single-parameter load usually considered in the ultimate load theory:

$$\mathbf{t} = t^\circ \hat{\mathbf{t}}_\mathbf{0} \tag{36}$$

Here t° is an unknown load factor and $\hat{\mathbf{t}}_\mathbf{0}$ is a prescribed reference load. Then the power of load

$$\dot{W} = \int_{S_T} \dot{\mathbf{u}}^T \mathbf{t} dS = t^\circ \int_{S_T} (\hat{\mathbf{t}}_\mathbf{0}^\mathbf{T} \dot{\mathbf{u}}) dS = t^\circ \nu^\circ \tag{37}$$

becomes linear with respect to the unknown load factor t° provided a power of reference load

$$\nu^\circ = \int_{S_T} \hat{\mathbf{t}}_\mathbf{0}^\mathbf{T} \dot{\mathbf{u}} dS \tag{38}$$

remains prescribed. In particular, we can normalise this power taking $\nu^\circ = 1$. This leads to the well known dual theorems concerning the ultimate load factor:

$$\text{find} \inf_{\dot{\boldsymbol{\lambda}}, \dot{\mathbf{u}}} \int_V \dot{\boldsymbol{\lambda}}^\mathbf{T} \boldsymbol{\kappa}_0 dV$$

subject to

$$\begin{aligned} &\mathbf{C_V} \dot{\mathbf{u}} - \mathbf{N} \dot{\boldsymbol{\lambda}} = \mathbf{0} \text{ in } V \\ &\dot{\boldsymbol{\lambda}} \geqslant \mathbf{0} \text{ in } V \\ &\int_{S_T} \hat{\mathbf{t}}_\mathbf{0}^\mathbf{T} \dot{\mathbf{u}} = 1 \text{ on } S_T \\ &\dot{\mathbf{u}} = \mathbf{0} \text{ on } S_U, \end{aligned} \tag{39}$$

$$\text{find} \sup_{\boldsymbol{\sigma}, t^\circ, \mathbf{t}} \int_{S_T} t^\circ dS$$

subject to

$$\begin{aligned} &\mathbf{N}^\mathbf{T} \boldsymbol{\sigma} \leqslant \boldsymbol{\kappa}_0 \text{ in } V \\ &\mathbf{C_V}^* \boldsymbol{\sigma} = \mathbf{0} \text{ in } V \\ &\mathbf{C}_\mathbf{S}^\mathbf{T} \boldsymbol{\sigma} = t^\circ \hat{\mathbf{t}}_\mathbf{0} \text{ on } S_T \\ &\mathbf{C}_\mathbf{S}^\mathbf{T} \boldsymbol{\sigma} - \mathbf{t} = \mathbf{0} \text{ on } S_U. \end{aligned} \tag{40}$$

Thus, the more we know about the desired distribution of surface load, the less restrictive are constrains on the displacement rates in the kinematic principle. Describing a priori the load up to a scalar factor gives us the most freedom for the displacement rates. They must merely produce fixed, e.g. unit, power of loading.

8. Conclusion

Summarising, we may say that only the analysis of linear elastic body under static and/or kinematic bilateral loading is governed by variational principles in the classical sense. All other problems lead to variational inequalities and, thus, are not reducible to systems of linear equations. The response of linear elastic bodies and elastic-strain hardening bodies to given load is unique, provided their constitutive matrices $\mathbf{E}^{-1}$ and $\mathbf{H}$ are positive definite.

The response of elastic-perfectly plastic body is unique only in terms of stresses and elastic strains. Moreover, the existence of solution is not warranted any more for arbitrary loading. If no a priori information regarding surface tractions is given, then the only way to find the response of a rigid-perfectly plastic body is to prescribe the displacement rates on its surface. Such problem can be interpreted also in terms of load optimisation.

If the direction of surface loading is prescribed, then the ultimate distribution of its modulus can be found, provided the power of loading is prescribed locally on the surface. Finally, when surface loading is known up to a scalar factor, the ultimate value of such factor follows from well known static and kinematic theorems.

The above mentioned results apply also for plastic bodies with smooth or piecewise-smooth non-linear yield surfaces. The critical feature is the convexity of such surface.

Appendix

Let column matrices $\mathbf{x}, \mathbf{b} \in R^m$ be related by a system of linear algebraic equations

$$\mathbf{Ax} + \mathbf{b} = \mathbf{0} \tag{A.1}$$

with symmetric positive definite $(m \times m)$-matrix of coefficients. Then the solution $\mathbf{x}_*$ of this system corresponds to the minimum point

$$L(\mathbf{x}_*) = \min_\mathbf{x} L(\mathbf{x}) \tag{A.2}$$

of the convex function

$$L(\mathbf{x}) = \frac{1}{2} \mathbf{x}^\mathbf{T} \mathbf{Ax} + \mathbf{x}^\mathbf{T} \mathbf{b} \tag{A.3}$$

This function is called a potential of system (A.1), since differentiating L with respect to $\mathbf{x}$ we obtain the left hand side of (A.1). The necessary and sufficient condition for $\mathbf{x}_*$ to be minimum of L is that the gradient of L must vanish at $\mathbf{x}_*$. Obviously, writing down this condition

$$\nabla \mathbf{L_x} = \mathbf{0} \tag{A.4}$$

we recover the initial system of Eq. (A.1).

Table A1
System of linear algebraic equations, its potential and equivalent QP-problems

System of equations:

$$\mathbf{A_{xx}x} + \mathbf{A_{xy}y} + \mathbf{b_x} = \mathbf{0}$$
$$\mathbf{A_{yx}x} + \mathbf{A_{yy}y} + \mathbf{b_y} = \mathbf{0}$$

Potential:

$$L(\mathbf{x},\mathbf{y}) = \frac{1}{2}\mathbf{x^T A_{xx} x} + \frac{1}{2}\mathbf{y^T A_{yy} y} + \mathbf{x^T A_{xy} y} + \mathbf{b_x^T x} + \mathbf{b_y^T y}$$

Saddle point:

$$L(\mathbf{x}_*,\mathbf{y}_*) = \min_{\mathbf{x}} \max_{\mathbf{y}} L(\mathbf{x},\mathbf{y})$$

Stationarity conditions:

$$\nabla \mathbf{L_x} = \mathbf{0}$$
$$\nabla \mathbf{L_y} = \mathbf{0}$$

Primal problem:

$$\text{find } \min_{x,y} \left\{ \frac{1}{2}\mathbf{x}^T \mathbf{A}_{xx}\mathbf{x} - \frac{1}{2}\mathbf{y}^T \mathbf{A}_{yy}\mathbf{y} + \mathbf{b}_x^T \mathbf{x} \right\}$$

subject to

$$\mathbf{A}_{yx}\mathbf{x} + \mathbf{A}_{yy}\mathbf{y} + \mathbf{b}_y = \mathbf{0}$$

Dual problem:

$$\text{find } \max_{x,y} \left\{ -\frac{1}{2}\mathbf{x}^T \mathbf{A}_{xx}\mathbf{x} + \frac{1}{2}\mathbf{y}^T \mathbf{A}_{yy}\mathbf{y} + \mathbf{b}_y^T \mathbf{y} \right\}$$

subject to

$$\mathbf{A}_{xx}\mathbf{x} + \mathbf{A}_{xy}\mathbf{y} + \mathbf{b}_x = \mathbf{0}$$

If matrix $\mathbf{A}$ were negative definite, then L would be concave and $\mathbf{x}_*$ would correspond to the maximum point of L. However, we are interested in linear systems that have slightly more complicated structure. Such system is written in the first row of Table A1. The overall matrix of coefficients remains symmetric but it has been subdivided now into sub-matrices that have different properties: $\mathbf{A_{xx}}$ is positive definite and $\mathbf{A_{yy}}$ is negative definite.

The result is that potential $L(\mathbf{x},\mathbf{y})$ defined in the second row of Table A1 becomes saddle-shaped: it is convex with respect to $\mathbf{x}$ and concave with respect to $\mathbf{y}$. The necessary and sufficient conditions

$$\nabla \mathbf{L_x} = \mathbf{0} \qquad \nabla \mathbf{L_y} = \mathbf{0} \tag{A.5}$$

for $(\mathbf{x}_*,\mathbf{y}_*)$ to be the saddle point of L coincide with the considered system of equations. Hence, point $(\mathbf{x}_*,\mathbf{y}_*)$ corresponds to the solution of this system.

Instead of looking for the saddle point we can solve a pair of constrained extremum problems, known as the dual problems. This circumstance is due to the fact that $(\mathbf{x}_*,\mathbf{y}_*)$ can be reached in two ways: either by minimising a certain convex function L' under constraints ensuring maximisation of L, or by maximising a certain concave function L'' under constraints ensuring minimisation of L. We expect functions L' and L'' to attain their extreme values at the point $(\mathbf{x}_*,\mathbf{y}_*)$. This requirement is met by the Legendre transforms [10]

$$L' = L - \mathbf{y^T} \nabla \mathbf{L_y} \qquad L'' = L - \mathbf{x^T} \nabla \mathbf{L_x}. \tag{A.6}$$

Taking advantage of them, we may write the dual problems as:

$$\text{find } \min_{\mathbf{x},\mathbf{y}} L' \quad \text{subject to} \quad \nabla \mathbf{L_y} = \mathbf{0} \tag{A.7}$$

$$\text{find } \max_{\mathbf{x},\mathbf{y}} L'' \quad \text{subject to} \quad \nabla \mathbf{L_x} = \mathbf{0}. \tag{A.8}$$

The last row of Table A1 gives explicit form of such problems.

Table A2
Linear complementarity problem, its potential and equivalent QP-problems

LCP-problem:

$$\begin{aligned}
&\mathbf{A_{xx}x} + \mathbf{A_{xy}y} + \mathbf{b_x} \geqslant \mathbf{0}\\
&\mathbf{A_{yx}x} + \mathbf{A_{yy}y} + \mathbf{b_y} \leqslant \mathbf{0}\\
&\mathbf{x} \geqslant \mathbf{0}, \mathbf{y} \geqslant \mathbf{0}\\
&\mathbf{x^T}\left(\mathbf{A_{xx}x} + \mathbf{A_{xy}y} + \mathbf{b_x}\right) = 0\\
&\mathbf{y^T}\left(\mathbf{A_{yx}x} + \mathbf{A_{yy}y} + \mathbf{b_y}\right) = 0
\end{aligned}$$

Potential:

$$L(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\mathbf{x^T A_{xx} x} + \frac{1}{2}\mathbf{y^T A_{yy} y} + \mathbf{x^T A_{xy} y} + \mathbf{b_x^T x} + \mathbf{b_y^T y}$$

Saddle point:

$$L(\mathbf{x}_*, \mathbf{y}_*) = \min_{\mathbf{x}\geqslant\mathbf{0}} \max_{\mathbf{y}\geqslant\mathbf{0}} L(\mathbf{x}, \mathbf{y})$$

Kuhn-Tucker conditions:

$$\begin{aligned}
&\nabla\mathbf{L_x} \geqslant \mathbf{0}\\
&\nabla\mathbf{L_y} \leqslant \mathbf{0}\\
&\mathbf{x} \geqslant \mathbf{0}, \mathbf{y} \geqslant \mathbf{0}\\
&\mathbf{x^T}\nabla\mathbf{L_x} = 0, \quad \mathbf{y^T}\nabla\mathbf{L_y} = 0
\end{aligned}$$

Primal problem:

$$\text{find } \min_{x,y}\left\{\frac{1}{2}\mathbf{x}^T\mathbf{A}_{xx}\mathbf{x} - \frac{1}{2}\mathbf{y}^T\mathbf{A}_{yy}\mathbf{y} + \mathbf{b}_x^T\mathbf{x}\right\}$$

subject to

$$\begin{aligned}
&\mathbf{A}_{yx}\mathbf{x} + \mathbf{A}_{yy}\mathbf{y} + \mathbf{b}_y \leqslant \mathbf{0}\\
&\mathbf{x} \geqslant \mathbf{0}
\end{aligned}$$

Dual problem:

$$\text{find } \max_{x,y}\left\{-\frac{1}{2}\mathbf{x}^T\mathbf{A}_{xx}\mathbf{x} + \frac{1}{2}\mathbf{y}^T\mathbf{A}_{yy}\mathbf{y} + \mathbf{b}_y^T\mathbf{y}\right\}$$

subject to

$$\begin{aligned}
&\mathbf{A}_{xx}\mathbf{x} + \mathbf{A}_{xy}\mathbf{y} + \mathbf{b}_x = \mathbf{0}\\
&\mathbf{y} \geqslant \mathbf{0}
\end{aligned}$$

Many interesting problems in physics and, in particular, in mechanics have internal structure of a linear complementarity problem (LC-problem) shown in the first row of Table A2. The potential function for this problem coincides with that for the set of equations discussed above. The formulation of the saddle point problem seems to be almost identical: one has merely to take into account the sign constraints on variables. However, the necessary and sufficient conditions for $(\mathbf{x}_*, \mathbf{y}_*)$ to be the saddle point are completely different:

$$\begin{aligned}
&\nabla\mathbf{L_x} \geqslant \mathbf{0}\\
&\nabla\mathbf{L_y} \leqslant \mathbf{0}\\
&\mathbf{x} \geqslant \mathbf{0}, \mathbf{y} \geqslant \mathbf{0}\\
&\mathbf{x^T}\nabla\mathbf{L_x} = 0\\
&\mathbf{y^T}\nabla\mathbf{L_y} = 0.
\end{aligned} \tag{A.9}$$

They are called Kuhn-Tucker conditions.

Legendre transforms (A.6) remain valid for the LC-problem and the dual problems read:

$$\begin{aligned}
&\text{find } \min_{\mathbf{x},\mathbf{y}} L'\\
&\text{subject to}\\
&\nabla\mathbf{L_y} \leqslant \mathbf{0}\\
&\mathbf{x} \geqslant \mathbf{0}
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
&\text{find } \max_{\mathbf{x},\mathbf{y}} L''\\
&\text{subject to}\\
&\nabla\mathbf{L_x} \geqslant \mathbf{0}\\
&\mathbf{y} \geqslant \mathbf{0}.
\end{aligned} \tag{A.11}$$

The explicit form of those problems is given in the last row of Table A2.

Until now we dealt with finite-dimensional spaces [11]. The above methodology can be generalised for linear topological spaces using the notions of variational inequalities and convex analysis [12, 13]. Limitations on the volume of the present paper preclude us from coming into details. We present, therefore, only the final result — the infinite-dimensional analogues of the dual models derived for a certain class of systems of linear partial differential equations (Table A3) and for LC-problems (Table A4).

For linear differential operators symmetry means self-adjointness. Therefore, we require $\mathbf{A_{xy}}$ and $\mathbf{A_{yx}}$ to be mutually adjoint. If $\mathbf{A_{xx}}$ and $\mathbf{A_{yy}}$ were differential operators, they should be self-adjoint. In our applications they happen to be algebraic, which requires the same properties as for the finite-dimensional case — the symmetry of both matrices plus positive definiteness of $\mathbf{A_{xx}}$ and negative definiteness of $\mathbf{A_{yy}}$.

In the stationarity conditions (A.5) simple derivatives should be replaced by sub-differentials. The result of such differentiation is a set and we are looking for the smallest (*infimum*) or largest element of this set (*supremum*). Finally, in order to evaluate the scalar product of two variables, we need to integrate over proper domain.

Table A3

System of linear partial differential equations, its potential and variational principles

System of equations:

$$\left.\begin{array}{l}\mathbf{A_{xx}x} + \mathbf{A_{xy}y} + \mathbf{b_x} = \mathbf{0}\\ \mathbf{A_{yx}x} + \mathbf{A_{yy}y} + \mathbf{b_y} = \mathbf{0}\end{array}\right\} \text{ in } V$$

boundary conditions on S

Potential:

$$L(\mathbf{x},\mathbf{y}) = \frac{1}{2}\int_V \mathbf{x^T A_{xx} x}\,dV + \frac{1}{2}\int_V \mathbf{y^T A_{yy} y}\,dV + \int_V \mathbf{x^T A_{xy} y}\,dV + \int_V \mathbf{b_x^T x}\,dV + \int_V \mathbf{b_y^T y}\,dV$$

$$+ \text{ boundary terms}$$

Saddle point:

$$L(\mathbf{x}_*,\mathbf{y}_*) = \inf_{\mathbf{x}} \sup_{\mathbf{y}} L(\mathbf{x},\mathbf{y})$$

Primal problem:

$$\text{find } \inf_{\mathbf{x},\mathbf{y}} \left\{ \frac{1}{2}\int_V \mathbf{x^T A_{xx} x}\,dV - \frac{1}{2}\int_V \mathbf{y^T A_{yy} y}\,dV + \int_V \mathbf{b_x^T x}\,dV + \text{boundary terms} \right\}$$

subject to

$\mathbf{A_{yx}x} + \mathbf{A_{yy}y} + \mathbf{b_y} = \mathbf{0}$ in V

boundary conditions on S

Dual problem:

$$\text{find } \sup_{\mathbf{x},\mathbf{y}} \left\{ -\frac{1}{2}\int_V \mathbf{x^T A_{xx} x}\,dV + \frac{1}{2}\int_V \mathbf{y^T A_{yy} y}\,dV + \int_V \mathbf{b_y^T y}\,dV + \text{boundary terms} \right\}$$

subject to

$\mathbf{A_{x}x} + \mathbf{A_{xy}y} + \mathbf{b_x} = \mathbf{0}$ in V

boundary conditions on S

Table A4
Linear complementarity problem, its potential and variational principles

LCP-problem:

$$\left.\begin{array}{c}\mathbf{A_{xx}x}+\mathbf{A_{xy}y}+\mathbf{b_x}\geqslant\mathbf{0}\\ \mathbf{A_{yx}x}+\mathbf{A_{yy}y}+\mathbf{b_y}\leqslant\mathbf{0}\\ \mathbf{x}\geqslant\mathbf{0},\mathbf{y}\geqslant\mathbf{0}\\ \int\limits_V\mathbf{x^T}\left(\mathbf{A_{xx}x}+\mathbf{A_{xy}y}+\mathbf{b_x}\right)dV=0\\ \int\limits_V\mathbf{y^T}\left(\mathbf{A_{yx}x}+\mathbf{A_{yy}y}+\mathbf{b_y}\right)dV=0\end{array}\right\}\text{ in }V$$

boundary conditions on S

Potential:

$$L(\mathbf{x},\mathbf{y})=\frac{1}{2}\int\limits_V\mathbf{x^T A_{xx} x}dV+\frac{1}{2}\int\limits_V\mathbf{y^T A_{yy} y}dV+\int\limits_V\mathbf{x^T A_{xy} y}dV+\int\limits_V\mathbf{b_x^T x}dV+\int\limits_V\mathbf{b_y^T y}dV+\text{boundary terms}$$

Saddle point:

$$L(\mathbf{x}_*,\mathbf{y}_*)=\inf_{\mathbf{x}\geqslant\mathbf{0}}\sup_{\mathbf{y}\geqslant\mathbf{0}}L(\mathbf{x},\mathbf{y})$$

Primal problem:

$$\text{find }\inf_{x,y}\left\{\frac{1}{2}\int\limits_V\mathbf{x}^T\mathbf{A}_{xx}\mathbf{x}dV-\frac{1}{2}\int\limits_V\mathbf{y}^T\mathbf{A}_{yy}\mathbf{y}dV+\int\limits_V\mathbf{b}_x^T\mathbf{x}dV\right\}$$

subject to

$$\left.\begin{array}{c}\mathbf{A}_{yx}\mathbf{x}+\mathbf{A}_{yy}\mathbf{y}+\mathbf{b}_y\leqslant\mathbf{0}\\ \mathbf{x}\geqslant\mathbf{0}\end{array}\right\}\text{ in }V$$

boundary conditions on S

Dual problem:

$$\text{find }\sup_{x,y}\left\{-\frac{1}{2}\int\limits_V\mathbf{x}^T\mathbf{A}_{xx}\mathbf{x}dV+\frac{1}{2}\int\limits_V\mathbf{y}^T\mathbf{A}_{yy}\mathbf{y}dV+\int\limits_V\mathbf{b}_y^T\mathbf{y}dV\right\}$$

subject to

$$\left.\begin{array}{c}\mathbf{A}_{xx}\mathbf{x}+\mathbf{A}_{xy}\mathbf{y}+\mathbf{b}_x\geqslant\mathbf{0}\\ \mathbf{y}\geqslant\mathbf{0}\end{array}\right\}\text{ in}V$$

boundary conditions on S

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