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LQG/LTR control of input-delayed discrete-time systems

D. HORLA[1]* and A. KROLIKOWSKI[2]

[1] Poznan University of Technology, Institute of Control, Robotics and Information Engineering, Piotrowo 3a, 60-965 Poznan, Poland
[2] Professor emeritus

Abstract. A simple robust *cheap* LQG control is considered for discrete-time systems with constant input delay. It is well known that the full loop transfer recovery (LTR) effect measured by error function $\Delta(z)$ can only be obtained for minimum-phase (MPH) systems without time-delay. Explicit analytical expressions for $\Delta(z)$ versus delay d are derived for both MPH and NMPH (nonminimum-phase) systems. Obviously, introducing delay deteriorates the LTR effect. In this context the ARMAX system as a simple example of noise-correlated system is examined. The robustness of LQG/LTR control is analyzed and compared with state prediction control whose robust stability is formulated via LMI. Also, the robustness with respect to uncertain time-delay is considered including the control systems which are unstable in open-loop. An analysis of LQG/LTR problem for noise-correlated systems, particularly for ARMAX system, is included and the case of proper systems is analyzed. Computer simulations of second-order systems with constant time-delay are given to illustrate the performance and recovery error for considered systems and controllers.

Key words: LQG control, loop transfer recovery, time-delay.

1. Introduction

The control of input-delayed systems has an abundant literature, see the below cited papers and the references therein. The same can be said about the robust LQG control based on LTR approach. However, there is still a space for research combining both control problems for various discrete-time systems.

The LQG/LTR control for discrete-time state-space systems was investigated for example in [30] where the general design aspects of LTR both at the input and the output of the control system are presented. In [28] the asymptotic case of LQG control, i.e. when the control weighting factor in the cost function tends to zero is considered for both predicting and filtering type of controller. The case of NMPH system is also discussed. Robust LQG/LTR control of discrete-time systems with time-delay at the input (or equivalently with computation delay) is a specific problem within a general LQG/LTR framework. In this context, some results are given in the literature, like for example: [13, 14, 20, 22, 33]. In [20], an extension to the case where the LQG/LTR problem with respect to the system input is solved for the square (i.e. when the number of inputs and outputs is equal) MPH system with d-sample delay is presented, where the recovery at both system input and output is investigated and the corresponding recovered loop transfer matrices are derived. Further extension of these results can be found in [33] where LQG/LTR problem was solved for NMPH systems with time-delays and explicit expressions of sensitivity and loop matrices are derived for the asymptotic behaviour of control system. From stability point of view, among the systems with delays two classes can generally be distinguished: *delay-dependent* stability and *delay-independent* stability. The former concerns the systems with unknown, time-varying (also randomly-varying) delays where the delay information, involved in the criterion, is directly used in the feedback controller. The stability conditions are usually derived in terms of LMIs with different levels of conservatism.

An example of *delay-dependent* stability criterion for LQ problem with variable input delay is given as an LMI in [32]. Its less conservative version is proposed in [11] or other in [4]. Other approaches to stabilization of delayed systems can be found in [5–8].

On the other hand, in the *delay-independent* case the delay information, not included in the criterion, is not directly used in the controller. In that case, the delay is usually assumed arbitrarily large but bounded integer. An example of using this stability criterion for stabilization of deterministic discrete-time state-space system is proposed in [35, 36] and in [17, 18], for system with structured uncertainty. In the case of constant and known delay the predictor-based controller can obviously assure the stabilization and performance recovery. If all eigenvalues of system matrix are inside or on the unit circle the system can be stabilized for arbitrarily large delay, so the open-loop system cannot be exponentially unstable.

The stability conditions in stochastic time-delay systems also fall into *delay-dependent* and *delay-independent* classes, where the former is generally less conservative. There are few methods to investigate the stabilization problem. One of them is based on the mean square BIBO stability sense like for example in [15, 16, 31], for both state and output-feedback controllers and possible nonlinear modelling errors.

*e-mail: dariusz.horla@put.poznan.pl

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In this note, the discrete-time Kalman filter-based *cheap* LQG control, i.e. with control weighting factor in the cost function equal to zero, for systems with constant delay at the input is considered. The concepts of LTR and delay-dependence are applied. Based on fundamental analysis of the recovery error for MPH and NMPH stochastic systems given in [20, 28, 33, 34], the original explicit expressions of error in function of input delay steps are derived. The resulting robustness with respect to uncertain delay for stable MPH, NMPH and unstable systems is analyzed and compared to state-feedback deterministic predictor-based control whose stability is determined by a given LMI condition [11]. In the case of unstable systems, the Kalman predictor-based LQG control in the so called *Smith predictor related structure* as proposed in [12], was additionally analyzed and compared with Kalman filter-based control.

The case of proper systems was analyzed with respect to the recovery using the transformation of proper system into its strictly proper model. Moreover, the usefulness of proper system models preventing the discontinuities of output signal at sampling instants, is analyzed showing that full recovery is not achievable.

The ARMAX model represented by the equivalent state-space model was taken as a simple example of stochastic noise-correlated system and analyzed with respect to the recovery. As a result, full recovery is shown for delay-free case $(d = 1)$. In other cases $(d > 1)$ there is no recovery, however the same nonzero LTR error occurs for both models.

Simulations were performed for the second-order systems with input delay. First, several computer tests were done for true system delay d and its model d_m used for implementation of considered controllers. Then, the heuristic dependence of closed-loop stability on the noise variance in the input-delayed ARMAX system is examined. The response of unstable system with given delay is presented, and finally, plots of magnitude of the error function for MPH and NMPH systems with different time-delays are included.

2. LQG/LTR for discrete-time systems with input delay

In this section the available preliminary results from literature are shortly surveyed and some new contributions are added. The considered discrete-time systems are described by the following state-space discrete-time SISO stochastic model

$$\underline{x}_{t+1} = F\underline{x}_t + \underline{g}u_{t-d+1} + \underline{w}_t\,, \tag{1}$$

$$y_t = \underline{h}^T\underline{x}_t + v_t\,, \tag{2}$$

where $\{\underline{w}_t\}$ and $\{v_t\}$ are sequences of independent random vector and scalar variables with zero mean and covariances $E\underline{w}_t\underline{w}_s^T = \Sigma_w\delta_{t,s}$, $Ev_tv_s = \sigma_v^2\delta_{t,s}$, respectively. The case $d = 1$ in (1) implies that system has a natural one-step delay in control channel and its transfer function is $G(z) = \underline{h}^T(zI-F)^{-1}\underline{g}$. The delay $d = 1, 2, 3, \ldots$ is given as multiplicity of sampling period T_s.

The system (1), (2) can be transformed to

$$\underline{x}^p_{t+1} = F\underline{x}^p_t + \underline{g}u_t + \underline{w}^p_t\,, \tag{3}$$

$$y_t = \underline{h}^T\underline{x}^p_{t-d+1} + v_t\,, \tag{4}$$

where $\underline{x}^p_t = \underline{x}_{t+d-1}$ and the Kalman filter estimate of $\underline{x}^p_t$ is given by

$$\underline{\hat{x}}^p_{t/t} = F^p\left[\underline{\hat{x}}^T_{t/t}, u_{t-d+1}, \cdots, u_{t-1}\right]^T, \tag{5}$$

where $F^p = \left[F^{d-1}, F^{d-2}\underline{g}, F^{d-3}\underline{g}, \cdots, F\underline{g}, \underline{g}\right]$ and the filtered estimate $\underline{\hat{x}}_{t/t}$ in terms of prediction $\underline{\hat{x}}_{t/t-1}$ follows from

$$\underline{\hat{x}}_{t/t} = \underline{\hat{x}}_{t/t-1} + \underline{k}_f\hat{y}^p_t\,, \tag{6}$$

where $\hat{y}^p_t = y_t - \underline{h}^T\underline{\hat{x}}_{t/t-1}$ is an output error. The Kalman predictor for $\underline{x}_{t+1}$ in steady-state is given by

$$\underline{\hat{x}}_{t+1/t} = F\underline{\hat{x}}_{t/t-1} + \underline{g}u_{t-d+1} + \underline{k}_p\hat{y}^p_t \tag{7}$$

and its gain is

$$\underline{k}_p = FP_f\underline{h}\left[\underline{h}^TP_f\underline{h} + \sigma_v^2\right]^{-1}, \tag{8}$$

where P_f is the solution of Riccati equation

$$P_f = FP_fF^T + \Sigma_w - FP_f\underline{h}\left[\underline{h}^TP_f\underline{h} + \sigma_v^2\right]^{-1}\underline{h}^TP_fF^T. \tag{9}$$

The filter gain is

$$\underline{k}_f = P_f\underline{h}\left[\underline{h}^TP_f\underline{h} + \sigma_v^2\right]^{-1}, \tag{10}$$

so $\underline{k}_p = F\underline{k}_f$ in view of (8) and (10). Finally, combining (6) and (7) one gets

$$\underline{\hat{x}}_{t/t-1} = F\underline{\hat{x}}_{t-1/t-1} + \underline{g}u_{t-d+1}\,. \tag{11}$$

The LQG *cheap* control law

$$u_t = \underline{k}_c^T\underline{\hat{x}}^p_{t/t} \tag{12}$$

aims to minimize the cost function

$$J = E\sum_{t=0}^{\infty} y_t^2, \tag{13}$$

so the optimal gain $\underline{k}_c$ is

$$\underline{k}_c^T = -\left[\underline{g}^TP_c\underline{g}\right]^{-1}\underline{g}^TP_cF \tag{14}$$

and P_c is the solution to Riccati equation

$$P_c = F^TP_cF - F^TP_c\underline{g}\left[\underline{g}^TP_c\underline{g}\right]^{-1}\underline{g}^TP_cF + Q. \tag{15}$$

In accordance with (13), the weighting matrix Q is taken as $Q = \underline{h}\underline{h}^T$. The control law (12) in view of (5) can be decomposed as follows

$$u_t = \underline{k}_c^T F^{d-1} \hat{\underline{x}}_{t/t} + \underline{k}_c^T \sum_{i=t-d+1}^{t-1} F^{t-i-1} \underline{g} u_i = u_t^f + u_t^d \quad (16)$$

where u_t^f is the part of control related to feedback from Kalman filter and $u_t^d = \underline{k}_c^T \sum_{i=t-d+1}^{t-1} F^{t-i-1} \underline{g} u_i$ is the part resulting from the number of delay steps.

2.1. Stable minimum-phase systems. The system (1), (2) is assumed to be stabilizable, detectable and MPH.

It can be shown e.g. in [28,30] that for $d = 1$ the gain $\underline{k}_c$ takes very simple form

$$\underline{k}_c^T = -\left(\underline{h}^T \underline{g}\right)^{-1} \underline{h}^T F \quad (17)$$

under the condition that $\underline{h}^T \underline{g} \neq 0$ i.e. the the *relative degree* r of the system transfer function $G(z)$ equals 1 (note that the transfer function $G(z)$ is determined for natural one step delay $d = 1$). This means that the corresponding transfer function of continuous-time system is strictly proper assuming zero-order hold discretization with sample period T_s. Discretizing the proper continuous-time systems, the *relative degree* r of $G(z)$ would be 0 and there would be a direct link between y_t and u_t in (2). This case is analyzed further in subsection 2.4. It is also worthy noting that if the *relative degree* of continuous-time transfer function is greater than 1 the excessive *discretization* zeros would be outside (or on) the unit circle.

The transfer function $G_f(z)$ of filter-based compensator defined by (6) and (12) can be manipulated into the form (see for example [28]),

$$G_f(z) = z\underline{k}_c^T \left[zI - \left(I - \underline{k}_f \underline{h}^T\right)\left(F + \underline{g}\underline{k}_c^T\right)\right]^{-1} \underline{k}_f = \\ = z\underline{k}_c^T \left[zI - F - \underline{g}\underline{k}_c^T\right]^{-1} \underline{k}_f, \quad (18)$$

where the right hand side is derived using (17) and $d = 1$. In that case the perfect recovery can take place, that is

$$\Delta(z) = \Xi(z) - G(z)G_f(z) = 0, \quad (19)$$

where the filter's loop transfer function $\Xi(z)$ is

$$\Xi(z) = \underline{h}^T \Phi(z) F \underline{k}_f \quad (20)$$

and $\Phi(z) = (zI - F)^{-1}$.

Time-delay in control channel of the system (1), (2) can alternatively be characterized by assuming that delay is incorporated into the transfer function $G(z)$ so that the new system model has the *Markov parameters* fulfilling the following properties

$$\underline{h}^T \underline{g} = \underline{h}^T F \underline{g} = \cdots = \underline{h}^T F^{d-2} \underline{g} = 0, \quad \underline{h}^T F^{d-1} \underline{g} \neq 0 \quad (21)$$

for $d \geq 1$; usually the notation $m_d = \underline{h}^T F^{d-1} \underline{g}$ is adopted. The transfer function of the new system model is $G_m(z) = z^{-(d-1)} G(z)$ and the *relative degree* r of $G(z)$ equals 1. In case when the *relative degree* of $G(z)$ is $r > 1$ then the *Markov parameters* fulfill (21) replacing d by r.

In [33], [20] it was shown that for MPH systems the error function $\Delta(z)$ for the optimal gain

$$\underline{k}_c^T = -m_d^{-1} \underline{h}^T F^d \quad (22)$$

has the form

$$\Delta(z) = \underline{h}^T \left(I - z^{-(d-1)} F^{d-1}\right) \Phi(z) F \underline{k}_f \quad (23)$$

for $d \geq 1$. In general $\Delta(z) \neq 0$, so the perfect recovery cannot be obtained except the case $d = 1$ where $\Delta(z) = 0$, what corresponds to (19). For $d \geq 2$ the error function is derived as

$$\Delta(z) = \underline{h}^T E(z) \underline{k}_f, \quad (24)$$

where the matrix $E(z)$ is given by

$$E(z) = \sum_{i=1}^{d-1} z^{-i} F^i$$

so it is a series which is bounded for stable matrix F and any value of time-delay. The error function $\Delta(z)$ can be interpreted as a *delay-dependent measure* of recovery degree in frequency domain.

As already mentioned, assuming that $G(z)$ has the *relative degree* $r \geq 2$, the *Markov parameters* correspond to (21) by taking $d = r$. The feedback gain follows then from (22) and the *relative degree* r is equivalent to the number of delay steps in the system.

2.2. Stable nonminimum-phase systems. It is known that if the system (1), (2) is NMPH then the perfect recovery is in general not possible. Similarly, it is interesting to see what happens when the LTR procedure is applied for this system with included time-delay. Usually, it is recommended to apply LTR for NMPH systems because the partial recovery could be then achieved [33], see [34] for continuous-time systems. The result for MPH systems can be modified for the NMPH systems after the appropriate factorization of $\Phi(z)$ [33]. For every NMPH system the *all-pass* factorization is possible

$$G(z) = \underline{h}^T \Phi(z) \underline{g} = G_a(z) G_{mph}(z) = \underline{h}_m^T G_a(z) \Phi(z) \underline{g}, \quad (25)$$

where $G_a(z)$ is *all-pass* and $G_{mph}(z)$ is MPH stable transfer function. Partial recovery ($\Delta(z) \neq 0$) for time-delayed system is then possible with LQG control gain

$$\underline{k}_c^T = -\left(\underline{h}_m^T F^{d-1} \underline{g}\right)^{-1} \underline{h}_m^T F^d, \quad (26)$$

where $\underline{h}_m$ can be easily obtained as a function of system parameters.

The recovery error is now

$$\Delta(z) = \left(\underline{h}^T - z^{-(d-1)} G_a(z) \underline{h}_m^T F^{d-1}\right) \Phi(z) F \underline{k}_f. \quad (27)$$

Again, in the case $d = 1$, the recovery error is

$$\Delta(z) = \left(\underline{h}^T - \underline{h}_m^T G_a(z)\right) \Phi(z) F \underline{k}_f, \tag{28}$$

so is not zero due to NMPH system.

Similarly to (24) one can derive from (27) that for $d \geq 2$

$$\Delta(z) = \underline{h}^T E(z) \underline{k}_f + (\underline{h}^T - \underline{h}_m^T G_a(z)) E_{nmph}(z) \underline{k}_f, \tag{29}$$

where the matrix $E_{nmph}(z)$ is given by

$$E_{nmph}(z) = \sum_{i=d}^{\infty} z^{-i} F^i$$

so compared with (24) an additional term appears which results from NMPH feature of the system.This term becomes unbounded for unstable matrix F.

By inspecting (27) it is easy to see [33], that the error function $\Delta(z)$ is identically zero, i.e. full recovery takes place, if the loop transfer function $\Xi(z)$ (20) to be recovered, satisfies the following conditions

- $\Xi(z) = G_a(z) \underline{h}_m^T \Phi(z) F \underline{k}_f$,
- $\underline{h}^T F \underline{k}_f = \underline{h}^T F^2 \underline{k}_f = \cdots = \underline{h}^T F^d \underline{k}_f = 0$.

This means that the observer loop has the same NMPH structure and at least as many delay steps as the system. If this is not fulfilled one may conclude in the light of above considerations that for stable both MPH and NMPH discrete-time state-space systems with input delay the recovery error $\Delta(z)$ is finite and depends explicitly on the system matrix and the value of delay, so for its exact calculation the delay d must be known.

2.3. Unstable systems. In general, there are no conditions preserving global stability of unstable systems with delay, so the existing techniques aim to find out the value of destabilizing delay d_{dest}. It is known that standard Smith predictor-based controllers are not suitable for unstable open-loop systems, however there are modifications such that unstable systems could be stabilizable, see as an example a scheme proposed in [26] for integrating and unstable systems described by SISO transfer functions.

An implementation form of LQG control which is free from unstable hidden modes was proposed in [12] for Kalman predictor-based controller $u_t = \underline{k}_c^T F^d \hat{\underline{x}}_{t/t-1}$. Adopting the result of [12], the final form for control law is

$$u_t = \underline{k}_c^T T(z) \underline{g} u_t + \underline{k}_c^T F^{d-1} \left(I + \Phi(z) \underline{k}_p \underline{h}^T\right)^{-1} \Phi(z) \times \times \left(\underline{k}_p y_t + z^{-d+1} \underline{g} u_t\right) = u_t^{fir} + u_t^p, \tag{30}$$

where $T(z) = (I - F^{d-1} z^{-d+1}) \Phi(z))$ is the finite response term and for implementation of control signal u_t, the finite response signal u_t^{fir} is calculated as follows

$$u_t^{fir} = \underline{k}_c^T \sum_{i=t-d+1}^{t-1} F^{t-i-1} \underline{g} u_i. \tag{31}$$

It can be seen that u_t^{fir} coincides with u_t^d in (16) for Kalman filter-based controller. The above approach can be interpreted as a modified control system of *Smith predictor structure* which is suited to cope with unstable systems. The *predictor* term u_t^p of control signal (30) is

$$u_t^p = \underline{k}_c^T F^d \left(I + \Phi(z) \underline{k}_p \underline{h}^T\right)^{-1} \Phi(z) \left(\underline{k}_p y_t + z^{-d+1} \underline{g} u_t\right). \tag{32}$$

On the other hand, the derivation of the *filter* term u_t^f in (16) yields

$$u_t^f = \underline{k}_c^T F^{d-1} \Big[\left(I + \underline{k}_f \underline{h}^T\right) \left(I + \Phi(z) \underline{k}_p \underline{h}^T\right)^{-1} \Phi(z) \times \times \left(\underline{k}_p y_t + z^{-d+1} \underline{g} u_t\right)\Big] + \underline{k}_f y_t \tag{33}$$

which is slightly different from the *predictor* term u_t^p given by (32).

The error function (24) is not suitable for unstable systems. Alternatively, the sensitivity approach can be applied by defining the sensitivity error function $\Delta_s(z) = \Delta_{so}(z) - \Delta_{sc}(z)$, where the observer sensitivity $\Delta_{so}(z) = [1 + \underline{h}^T \Phi F \underline{k}_f]^{-1}$ and the closed-loop sensitivity $\Delta_{sc}(z) = [1 + G(z) G_f(z)]^{-1}$ are both stable. Similarly to (24), the error function $\Delta_s(z)$ can be manipulated to $\Delta_s(z) = \underline{h}^T E_s(z) \underline{k}_f$ where the matrix $E_s(z)$ depends on system parameters, so the value $\max_\omega |\Delta_s(z)|$ might be a good measure of recovery.

2.4. Proper systems. Taking care of discontinuity of the system output at sampling instants [1], the model (1), (2) takes a form

$$\underline{x}_{t+1} = F \underline{x}_t + \underline{g} u_{t-d+1} + \underline{w}_t, \tag{34}$$

$$y_{t+1} = \underline{h}^T \underline{x}_{t+1} + e u_{t-d} + v_t, \tag{35}$$

where $e \geq 0$ and the model transfer function $G_m(z) = G(z) z^{-(d-1)} + e z^{-d}$ is causal and strictly proper. The transfer function $G(z)$ can be considered as a nominal model. The Riccati equation takes now the form of generalized equation (15), i.e.

$$P_c = F^T P_c F - \left(F^T P_c \underline{g} + e \underline{h}\right) \left[\underline{g}^T P_c \underline{g} + e^2\right]^{-1} \times \times \left(\underline{g}^T P_c F + e \underline{h}^T\right) + Q \tag{36}$$

and assuming $d = 1$, the feedback gain is

$$\underline{k}_c^T = -\left[\underline{g}^T P_c \underline{g} + e^2\right]^{-1} \left(\underline{g}^T P_c F + e \underline{h}^T\right). \tag{37}$$

The control law follows then from (36), (37) for $P_c = 0$ what requires the weighting matrix $Q = \underline{h}\underline{h}^T$. This gives

$$\underline{k}_c^T = -e^{-1} \underline{h}^T. \tag{38}$$

It is easy to see that the same feedback gain $\underline{k}_c^T$ can be deduced directly from (35), however for implementing the causal control law $u_t = \underline{k}_c^T \hat{\underline{x}}_t$ the Kalman filter

$$\hat{\underline{x}}_{t+1} = F \hat{\underline{x}}_t + \underline{g} u_t + \underline{k}_f (y_{t+1} - \hat{y}_{t+1}) \tag{39}$$

is used, where $\hat{y}_{t+1} = \underline{h}^T \hat{\underline{x}}_{t+1} + eu_t$ and the corresponding Kalman filter gain is as in (10).

To check the recovery, the compensator transfer function G_f $(y_t \to u_t)$ derived using (39), and for $d = 1$, gives the following form

$$G_f(z) = z\underline{k}_c^T \left[z\left(I + \underline{k}_f \underline{h}^T\right) - \left(F + \underline{g}\underline{k}_c^T\right) + e\underline{k}_f \underline{k}_c^T\right]^{-1} \underline{k}_f. \quad (40)$$

Obviously, the above transfer function differs essentially from (18). In fact, the term inside square brackets in (40) is a *matrix pencil* and solution of its eigenvalue problem is a difficult numerical task. Moreover, taking into account $G_m(z) = G(z) + ez^{-1}$, one may see that the term $G_m(z)G_f(z)$ makes the recovery analysis of (19) complicated.

For comparison, a direct application of Kalman filter based on (6), (7), to system (34), (35), is considered taking $y_{t+1} - \hat{y}_{t+1}$ as an input to the filter where $\hat{y}_{t+1} = \underline{h}^T \hat{\underline{x}}_{t+1/t} + eu_t$ and the corresponding Kalman filter gain is as in (10). Then the transfer function of the compensator can be derived as

$$G_f(z) = z\underline{k}_c^T \left[zI - \left(I - \underline{k}_f \underline{h}^T\right)\left(F + \underline{g}\underline{k}_c^T\right) + e\underline{k}_f \underline{k}_c^T\right]^{-1} \underline{k}_f \quad (41)$$

which shows the difference from (40) however, the above form of $G_f(z)$ corresponds to (18). Specifically, for $e = 0$, the above transfer function is equivalent to (18), and recovery analysis is feasible for $e > 0$.

The LTR problem was also considered in [19] however, the loop recovery was analyzed based on the sensitivity matrix at the system input. The skewed sampling model was considered with output sampling equation taken in the form

$$y_{t+1} = \underline{h}^T \underline{x}_t + eu_t + v_t, \quad (42)$$

where $e \geq 0$ and the model transfer function is $G_m(z) = z^{-1}(G(z) + e)$ which is strictly proper. The corresponding Riccati equation is given by (36), and the feedback gain given by (37). In this case the control signal is also given by (38).

For state estimation the Kalman filter (39) is used with $\hat{y}_{t+1} = \underline{h}^T \hat{\underline{x}}_t + eu_t$ and the corresponding Kalman filter gain $\underline{k}_f$ is given by (10).

Obviously, the both models, i.e. (34), (35) and (42), (34) prevent discontinuity of output signal at sampling instants, however from LTR point of view the model (34), (35) is less useful. On the other hand, in the case of delay-free ($d = 1$), and MPH skewed sampling model (42), (34), the recovery, on the basis of the result given in subsection 2.1, is possible.

In [19] it was shown that if $G_m(z)$ is MPH then the sensitivity matrix at the input converges asymptotically to the sensitivity matrix at the system input under perfect observation. This confirms the convergence of the LTR procedure for MPH case. For NMPH systems, even for delay-free case the full recovery is not attainable.

It is known [23] that the cost function (13) is minimized if and only if the cost $J_r = E\sum_{t=0}^{\infty} y_{t+r}^2$ is minimized where r is the *relative degree*. This means that models described by output equations (35), (42) do not deteriorate the performance of control given by the feedback gain (38).

The compensator transfer function for the skewed sampling model, assuming $d = 1$, has the following form

$$G_f(z) = z\underline{k}_c^T \left[zI - \left(F + \underline{g}\underline{k}_c^T\right) + \underline{k}_f \underline{h}^T + e\underline{k}_f \underline{k}_c^T\right]^{-1} \underline{k}_f. \quad (43)$$

One may observe that for $\underline{k}_c^T$ given by (38) the above transfer function coincides with right hand part of (18). Taking into account $G_m(z) = z^{-1}(G(z) + e)$, one may see that this is not the case of full recovery in the sense of (19) because $\Delta \neq 0$.

It is worthy noting that different forms of compensator transfer functions (40), (41), (43) come from different Kalman filter corresponding to the particular model $G_m(z)$ of a proper system.

Alternatively, in order to make use of the results from subsection 2.1 applied to strictly proper system (1), (2), and assuming $d = 1$, the following extended model

$$\underline{x}_{t+1}^e = F^e \underline{x}_t^e + \underline{g}^e r_t + \underline{w}_t^e, \quad (44)$$

$$y_{t+1} = \underline{h}^{eT} \underline{x}_t^e + v_t^e, \quad (45)$$

can be adopted, where $\underline{x}_t^e = \left(\underline{x}_t^T, u_t\right)^T$ is an extended state vector and $F^e = \begin{bmatrix} F & \underline{g} \\ \underline{0}^T & -1 \end{bmatrix}$, $\underline{g}^e = (\underline{0}^T, 1)^T$, $\underline{h}^{eT} = (\underline{h}^T, e)$ and the optimal controls follow from recursion $u_{t+1} = -u_t + r_t$, this means that control signal u_t can be retrieved from v_t. It is worthy noting that equation (45) corresponds to output sampling equation (42). The Riccati equation and control gain are analogous to (15), (14), so the matrix solution to Riccati equation is now $P_c^e = Q = \underline{h}^e \underline{h}^{eT}$, moreover, the condition $\underline{h}^{eT} \underline{g}^e > 0$ is fulfilled.

Systems (34), (35) and (44), (45) are, for a given initial condition, equivalent from control point of view. Moreover, the dynamics of the Kalman filter remains the same, however the question is whether equality (19) will hold, even for delay $d = 1$.

To show the difference between both cases the feedback gain for strictly proper extended model (44), (45) with $e > 0$, is derived. It is known that the control Riccati equation is now

$$\underline{k}_c^{eT} = -\left(e^{-1}\underline{h}^T F, e^{-1}\underline{h}^T \underline{g} - 1\right), \quad (46)$$

where decomposition $\underline{k}_c^{eT} = \left(\underline{k}_{cx}^{eT}, k_{cu}^e\right)$ is used. Noting the following property of the extended model with nonsingular matrix F

$$k_{cu}^e - \underline{k}_{cx}^{eT} F^{-1} \underline{g} - 1 = 0, \quad (47)$$

one may conclude that

$$\underline{k}_c^T = \underline{k}_{cx}^{eT} F^{-1}, \quad (48)$$

that corresponds to (38). In the special case of $e = \underline{h}^T \underline{g}$, the gain $\underline{k}_c^{eT} = (\underline{k}_c^T, 0)$ where $\underline{k}_c^T$ is given by (17), and the recovery takes place as described in subsection 2.1.

The following transfer functions for extended model can be found:

- $G_m(z) = (G(z)+e)(z+1)^{-1}$ when r_t is an input or $G_m(z) = G(z)+e$ when u_t is an input,
- $\Xi_m(z) = \Xi(z)$,
- and the compensator transfer function $G_f(z)$

$$\begin{aligned}G_f(z) &= z\underline{k}_{cx}^{eT}\left[zI-\left(I-\underline{k}_{fx}^{e}\underline{h}^T\right)\left(F+\underline{g}\underline{k}_{cx}^{eT}(z+1)+\right.\right.\\&\quad\left.\left.+\ e\underline{k}_{fx}^{e}\underline{k}_{cx}^{eT}\right)\right]^{-1}\underline{k}_{fx}^{e} =\\&= z\underline{k}_{c}^{T}\left[zI-\left(I-\underline{k}_{f}\underline{h}^T\right)\left(F-e^{-1}(z+1)\underline{g}\underline{h}^T-\right.\right.\\&\quad\left.\left.-\ \underline{k}_{f}\underline{h}^T\right)\right]^{-1}\underline{k}_{f}\,.\end{aligned}\tag{49}$$

The last term in (49) is derived for $\underline{k}_c^T$ given by (38) and $e > 0$. Moreover, the following notations are used: $k_{cu}^e = k_{fu}^e = 0$, with the decomposition $\underline{k}_f^e = \left[\underline{k}_{fx}^{eT}, k_{fu}^e\right]^T$. This means that $\underline{k}_f = \underline{k}_{fx}^e$ and $\underline{k}_c^T = \underline{k}_{cx}^{eT}$.

Obviously, full recovery is only possible for $e = 0$, then noting that $r_t = (z+1)u_t$, the transfer function $G_f(z)$ (49) coincides with $G_f(z)$ in (18).

Hence, the recovery in the sense of (19) is not possible for any $e > 0$, however, for small enough e the following holds

$$\Delta(z) = \Xi(z) - (G(z)+e)G_f(z) = -eG_f(z) \approx 0. \tag{50}$$

From above equation one can consider the model (44), (45) as most suitable for analysis of recovery error w.r.t. parameter e.

To analyze the case of any delay $d \geq 1$ the formula (22) was used for extended model (44), (45) including delay d. The resulting control gain

$$\begin{aligned}\underline{k}_c^{eT} &= -\left(\underline{h}^T\sum_{i=0}^{d-2}F^i\underline{g}(-1)^{d-2-i}+e(-1)^{d-1}\right)^{-1}\times\\&\quad\times\left(\underline{h}^TF^d,\ \underline{h}^T\sum_{i=0}^{d-1}F^i\underline{g}(-1)^{d-1-i}+e(-1)^{d}\right)\end{aligned}\tag{51}$$

is not asymptotically convergent as $d \to \infty$, because of the eigenvalue -1 of matrix F^e. It is worthy noting that for $d = 1$ the above formula is the same as (46), and additionally, if $e = 0$ then it coincides with (17).

2.5. Stability comments. The *delay-dependent* LMI conditions for robust stability of noise-free system (1) with unknown time-varying delay d_t belonging to the interval $d_l \leq d_t \leq d_u$ are given in [32] and [11] where the lower d_l and upper d_u bounds are known. The system considered in [11] is under state-feedback prediction-based controller $u_t = \underline{k}_c^T\hat{\underline{x}}_{t+h/t}$ and compared to the time-delayed state feedback controller $u_t = \underline{k}_c^T\underline{x}_{t-d_t}$ with a given gain $\underline{k}_c$ and a given prediction horizon h.

This approach is adopted for comparison study when the unknown delay d is constant, i.e. $d_l = d_u$, and $\hat{\underline{x}}_{t+h/t}$ with horizon h can be obtained on the base of (7) replacing d by h. The stability criterion given as LMI is used to determine the maximum achievable delay d_{dest} that guarantees stability for unstable deterministic system – for some positive small enough scalars ε_1, ε_2 as the tuning parameters, see Corollary 2 in [11]. This delay value can serve as an indication on what the value of d_{dest} in corresponding stochastic system, can be.

To be specific, consider the ARMAX model given by

$$y_t = G(z^{-1})u_{t-d+1} + G_e(z^{-1})e_t\,, \tag{52}$$

where $G(z^{-1}) = \dfrac{B(z^{-1})}{A(z^{-1})}$, $G_e(z^{-1}) = \dfrac{C(z^{-1})}{A(z^{-1})}$, and at the same time $G(z) = \underline{h}^T(zI-F)^{-1}\underline{g}$, $G_e(z) = \underline{h}^T(zI-F)^{-1}\underline{k}_e + 1$ with $A(z^{-1}), B(z^{-1})$ and $C(z^{-1})$ polynomials in the operator z^{-1}, i.e. $A(q^{-1}) = 1 + a_1z^{-1} + \ldots + a_nz^{-n}, B(z^{-1}) = b_1z^{-1} + \ldots + b_nz^{-n}, C(z^{-1}) = 1 + c_1z^{-1} + \ldots + c_nz^{-n}$ and $\{e_t\}$ assumed to be a sequence of independent variables with zero mean and variance σ_e^2.

Model (52) has an equivalent state-space representation,

$$\underline{x}_{t+1} = F\underline{x}_t + \underline{g}u_{t-d+1} + \underline{k}_e e_t\,, \tag{53}$$

$$y_t = \underline{h}^T\underline{x}_t + e_t\,, \tag{54}$$

where $\underline{g} = [b_1, \ldots, b_n]^T$, $\underline{k}_e = [c_1 - a_1, \ldots, c_n - a_n]^T$, $\underline{h}^T = [1, 0, \ldots, 0]$ and $F = \begin{bmatrix} -\underline{a} & \begin{matrix} I_{n-1} \\ \underline{0}^T \end{matrix} \end{bmatrix}$ with $\underline{a} = [a_1, \ldots, a_n]^T$.

The above model can be transformed to a model with uncorrelated noise. To get this the augmented vector $\underline{p}_t = \left[\underline{x}_t^T, y_t\right]^T$ is introduced which allows for the new state-space equation

$$\underline{p}_{t+1} = \tilde{F}\underline{p}_t + \tilde{\underline{g}}u_{t-d+1} + \underline{\theta}_t\,, \tag{55}$$

where $\tilde{F} = EA^*$, $\tilde{\underline{g}} = E\left[\underline{g}^T, 0\right]^T$, $\underline{\theta}_t = E\left[\underline{0}^T, e_{t+1}\right]^T$ and $E = \begin{bmatrix} I & \underline{0} \\ \underline{h}^T & 1 \end{bmatrix}$, $A^* = \begin{bmatrix} F^* & \underline{k}_e \\ \underline{0}^T & 0 \end{bmatrix}$. Now one can formulate the following proposition: the stability criterion for the closed-loop stability of ARMAX system with Kalman filter-based controller has the same form as LMI condition given in [11] for predictor-based controller in the case of deterministic system.

To show this, it is enough to derive the predictor form from (55) and replace d by h, so the result is

$$\hat{\underline{p}}_{t+h} = \tilde{F}^h\underline{p}_t + \sum_{i=0}^{h-i-1}\tilde{F}^{h-i-1}\tilde{\underline{g}}u_{t-h+1}. \tag{56}$$

It is easy to see that the predictor (56) has the same form as in [11] and together with the controller $u_t = \tilde{\underline{k}}_c^T\hat{\underline{p}}_{t+h/t}$ constitute the base for derivation of LMI condition that is formulated as follows: for any $\tilde{\underline{k}}_c$ such that the matrix $\tilde{F} + \tilde{\underline{g}}\tilde{\underline{k}}_c$ is Hurwitz, and for $d_l = d_u = h$, there exist a feasible solution to a given LMI.

As mentioned earlier, the stabilizable and detectable system with arbitrarily large delay in the control input can be asymptotically stabilized by either linear state or output feedback as long as the open-loop system is not asymptotically unstable [25].

Related result is given in [10] (refer also to [9, 27]), where it is proven that the achievable *delay margin* (corresponding to d_{dest} in this note) for system $G(z)$ and stabilizing controller is strictly greater than zero if and only if the system $G(z)$ has no negative real unstable poles. In particular, the system has *delay margin* equal to zero if and only if the system matrix F has a real unstable pole at $(-\infty,-1]$.

For illustration consider an example of simple unstable open-loop system analyzed in [25],

$$x_{t+1} = fx_t + u_{t-d+1}$$

for $f > 1$ and delay $d = 1,2,\ldots$ with controller $u_t = k_c x_t$. Using the root locus method it was shown that for large enough delay d the closed-loop system is not asymptotically stable for any choice of k_c. Taking into account the controller (22) one gets $k_c = -f$ for any delay d, and the following characteristic equation

$$z^d - fz^{d-1} + f = 0$$

for closed-loop system. It is easy to verify that except the delay-free case $d = 1$ when the closed-loop pole is $z = 0$ for any $f > 1$, the closed-loop system is unstable for any $d \geq 2$. Concluding, the controller (22) is not suitable for this unstable open-loop system for every time-delay $d \geq 2$.

The additive uncertain system with input time-delay and possible unstable poles was considered in [21], where it was shown that achievable robustness margin decreases to zero as the time-delay value increases.

3. LTR for ARMAX model

ARMAX model given by equations (53), (54) can be regarded as a special case of noise-correlated stochastic state-space models [3]. These equations can take the following representation

$$\underline{x}_{t+1} = F^* \underline{x}_t + \underline{g} u_{t-d+1} + \underline{k}_e y_t\,, \tag{57}$$

$$y_t = \underline{h}^T \underline{x}_t + e_t, \tag{58}$$

where $F^* = F - \underline{k}_e \underline{h}^T$ and $\underline{k}_p = \underline{k}_e$ as an equivalence condition between state-space (1), (2) and ARMAX (53), (54) models. In the considered steady-state case, the Kalman filter $\hat{\underline{x}}_{t/t}$ and the Kalman predictor $\hat{\underline{x}}_{t/t-1}$ estimates derived from (57), (58) coincide asymptotically, i.e. $\hat{\underline{x}}_{t/t-1} \to \hat{\underline{x}}_{t/t} \to \hat{\underline{x}}_t$. The Kalman filter equation takes then a simple form

$$\hat{\underline{x}}_{t+1} = F^* \hat{\underline{x}}_t + \underline{g} u_{t-d+1} + \underline{k}_e y_t\,. \tag{59}$$

In order to use the LQG approach to ARMAX model (52), one possible way is to replace the polynomial $B(z^{-1})$ and vector $\underline{g}$ by $(n-d+1)$-th order polynomial $B_d(z^{-1})$ and n-th vector $\underline{g}_d = [b_d,\ldots,b_{n-d+1},0\ldots0]^T$, respectively and neglecting the delay in control channel.

Then, for the model equivalent to (53), (54), the optimal control law has the following form [2,24]

$$u_t = -\left(\underline{g}_d^T S_d \underline{g}_d\right)^{-1} \underline{g}_d^T S_d \left(F^* \hat{\underline{x}}_t + \underline{k}_e y_t\right), \tag{60}$$

where $S_d = \sum_{j=0}^{d-1} F^{jT} \underline{hh}^T F^j$ (for MPH systems) and $\hat{x}_t$ follows from (59). Again, it is assumed that the delay d in system (57), (58) is characterized by the *Markov parameters* (21).

Using S_d, m_d in (60) and taking into account (59), the controller transfer function can be derived as follows

$$G_f(z) = m_d^{-1}\Big\{\underline{h}^T F^{d-1} F^* \left[zI - G_d^* F^*\right]^{-1} \cdot \cdot\, G_d^* \left(I + \underline{g}_d \underline{h}^T F^{d-1}\right) \underline{k}_e\Big\}, \tag{61}$$

where $G_d^* = I - m_d^{-1} \underline{g}_d \underline{h}^T F^{d-1}$.

Particularly, for $d = 1$, from (60), u_t reads

$$u_t = -m_1^{-1} \underline{h}^T \left(F^* \hat{\underline{x}}_t + \underline{k}_e y_t\right). \tag{62}$$

Again, assuming $d = 1$ in system (1), (2) with the optimal control law (12), (17), it is easy to see that asymptotically, the filter equation derived from (7) is the same as (59), so the considered control law coincides with (62). Moreover, the filter's loop transfer function $\Xi(z)$ is the same. This means that in MPH ARMAX system the full recovery is attainable despite the noise correlation. One may conclude that LTR property does not depend on the particular model amongst the covariance equivalent ones. Generally, this is not the case for any d, however the transfer functions of MV ARMAX and LQG state-space controllers are identical.

It is interesting to remark that the control law (60) can also be obtained using the uncorrelated-noise system (55), i.e. solving the standard LQG problem for this system and replacing $\underline{x}_t$ by $\hat{\underline{x}}_t$.

Additionally, one can remark that the optimal gain (22) obtained for uncorrelated-noise system can also be derived from (60) by substituting S_d, putting $\underline{k}_e = 0$ and taking in account the *Markov parameters* (21).

Obviously, the advantage of this modeling is the simple implementation of LQG control without need of explicit solution to Riccati equation.

4. Simulation study

After ZOH discretization of three different second-order continuous-time systems, the following discrete-time models $G_i(z^{-1})z^{-d+1} = \dfrac{B_i(z^{-1})}{A_i(z^{-1})} z^{-d+1}$, $i = 1,2,3$, in z^{-1} operator, with time-delay d are obtained for sampling period $T_s = 0.5$:

- stable MPH system $G_1(z^{-1})$ with
  $B_1(z^{-1}) = -0.3262z^{-1} - 0.1224z^{-2}$,
  $A_1(z^{-1}) = 1 - 0.8297z^{-1} + 0.1535z^{-2}$,
- stable NMPH system $G_2(z^{-1})$ with
  $B_2(z^{-1}) = -0.1612z^{-1} + 0.2856z^{-2}$,
  $A_2(z^{-1}) = 1 - 0.9744z^{-1} + 0.223z^{-2}$
  having one NMPH zero at 1.772, an all-pass transfer function $G_a(z) = \dfrac{z - 1.772}{1 - 1.772z}$ and according to (25) and (26) $\underline{h}_m^T = (0.5452, 1.3077)$, $\underline{k}_c^T = (-0.8391, -1.9091)$,

- unstable system $G_3(z^{-1})$ with
  $B_3(z^{-1}) = 1.352z^{-1} - 0.439z^{-2}$,
  $A_3(z^{-1}) = 1 - 5.088z^{-1} + 2.718z^{-2}$.

Other systems can be constructed as follows $G_{i,j}(z^{-1})z^{-d+1} = \dfrac{B_i(z^{-1})}{A_j(z^{-1})}z^{-d+1}$, $i, j = 1,2,3$ for $i \neq j$ and $G_i(z^{-1}) = G_{i,i}(z^{-1})$ for $i = j$.

System parameters needed for simulation calculations are obtained from given above transfer functions, according to representation of (52).

In computer tests different configurations of system delay d and its model d_m taken for the controller implementation were tested. In other words the under-modeling $d_m < d$ and over-modeling $d_m > d$ cases were simulated.

Obviously, LQG/LTR method with controllers (22), (26) as well as LMI approach [11] ensure stability for all under- and over-modeling configurations of time-delays in case of stable systems both MPH and NMPH (in particular $G_1(z^{-1}), G_2(z^{-1})$). For unstable systems (in particular $G_3(z^{-1})$) the global closed-loop stability can not be assured even in case of perfect delay matching $d = d_m$, however certain d_{dest} can be established also in case of stochastic system characterized by the noise variance σ_e^2 but the control system is not robust because the stability can be obtained only if there is no modelling error.

As already mentioned in section 2.5, the determination of the delay d_{dest} by means of the LMI stability condition given in [11] holds for noise-free deterministic system under the prediction-based controller with feedback gain $\underline{k}_c$ (14). This value of the delay may give an indication on what the value of d_{dest} in corresponding stochastic system can be. Thus, it is supposed that the value of d_{dest} in deterministic case can be regarded as a bound of d_{dest} in corresponding stochastic case when noise variance tends to zero for any delay $d > 1$.

Calculations based on the considered LMI condition for the unstable system G_3 were performed. For scalars $\varepsilon_1 = \varepsilon_2 = 10^{-6}$ (tuning parameters), the obtained value of destabilizing time-delay is $d_{\text{dest}} = 14$.

In Fig. 1, the plots of d_{dest} versus variance σ_e^2 are shown for unstable systems G_{13}, G_{23} for MPH and NMPH cases with controllers (22) and (26), respectively. It was found that for $\sigma_e^2 = 0$ the obtained average values are $d_{\text{dest}} = 23, 22$ for MPH and NMPH case, respectively. This shows the conservatism of the LMI approach. Obviously, in both cases the stability occurs only for perfect modeling $d = d_m$.

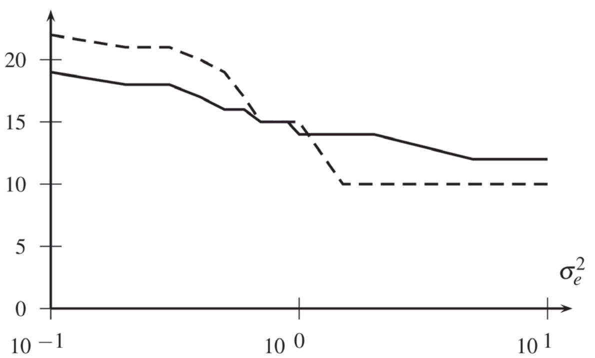

Fig. 1. Destabilizing values d_{dest} versus noise variance $\sigma_e^2 \geq 10^{-1}$, solid line MPH, dashed line NMPH.

Plot of output and control variables for unstable G_3 MPH noise-free system with non-zero initial conditions, $d = d_m = 14$ and for controller (22), is given in Fig. 2.

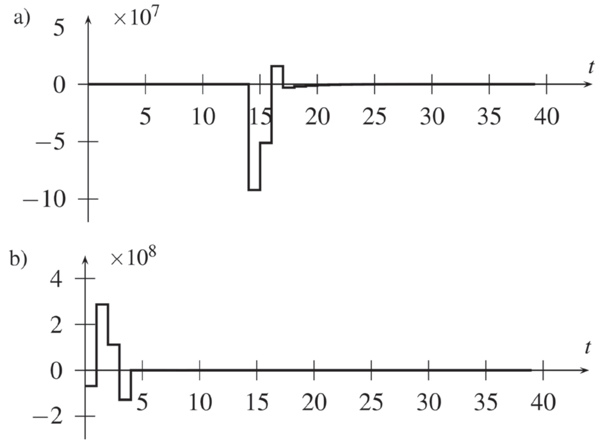

Fig. 2. Output and control variables, a) output, b) control, G_3 MPH, $d = d_{\text{m}} = 14$.

Figs. 3, 4 present plots of singular values of the error function $\Delta(z)$ in frequency domain over the range $\left(0, \frac{\pi}{T_s}\right)$ for time-delays

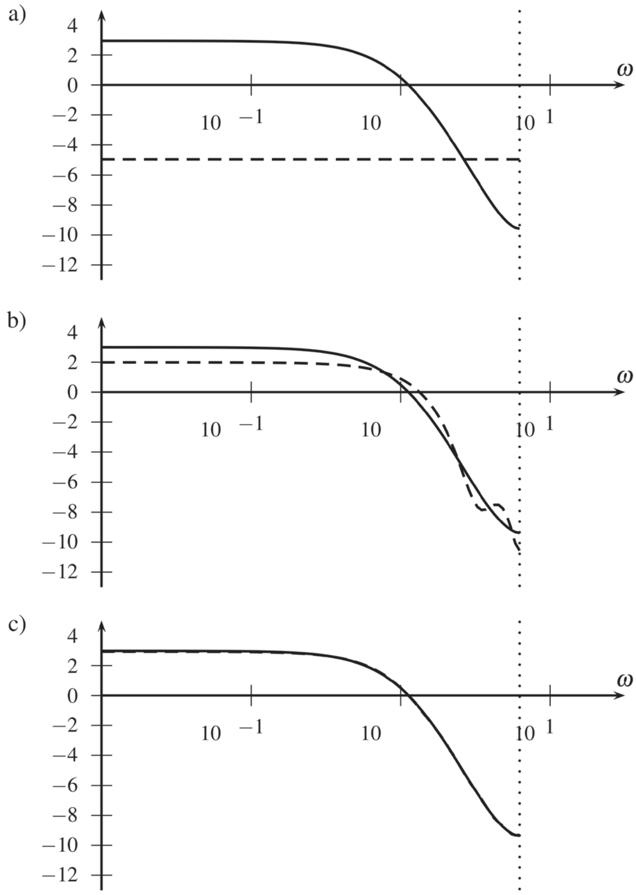

Fig. 3. Singular values [dB] a) $d = 3$, b) $d = 6$, c) $d = 11$, solid line MPH G_1, dashed line NMPH G_{21}.

$d = d_m = 2, 5, 10$, where $T_s = 0.5$. The plots for systems $G_1(z^{-1})$ and $G_{21}(z^{-1})$ and controllers (22), (26), respectively are shown in Fig. 3. The corresponding plots for $G_2(z^{-1})$ compared to $G_{12}(z^{-1})$ are shown in Fig. 4. Note that the system matrix F in both cases remains unchanged.

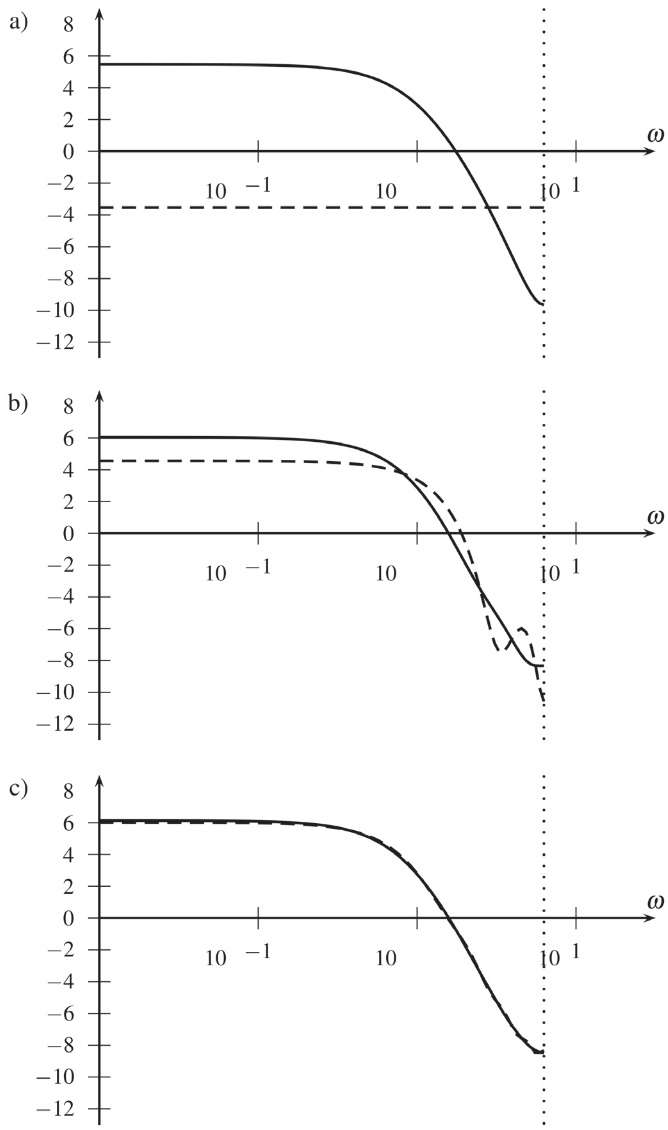

Fig. 4. Singular values [dB] a) $d = 3$, b) $d = 6$, c) $d = 11$, solid line MPH G_{12}, dashed line NMPH G_2.

5. Conclusions

LQG control of discrete-time SISO system with delayed control in the context of LTR effect is presented and shortly surveyed. The analytical expressions for recovery error function for stable MPH and NMPH systems with constant input delay are given.

Finding destabilizing value of time-delay in case of unstable open-loop system is another important question. To this end, second-order unstable MPH and NMPH stochastic ARMAX systems were taken into consideration and the results compared with the result based on LMI robust stability condition [11] for corresponding deterministic system. This was done by simulation tests for various delays using LQG controllers (22) and (26) for ARMAX model. The case of more general delayed stochastic systems ensuring the mean-square stability needs more research. The problem with discontinuity of output signal at sampling instants in case of proper systems needs special modelling. In this regard two models: the left side limit sampling model [1] as well as skewed sampling model [19] are considered presenting the corresponding compensator transfer functions which are much different from transfer function of strictly proper system. In this respect using of classical Kalman filter allows for improvement. In general, there is a lack of full LTR property for proper systems.

Specifically, the control laws with LTR feature for both input-delayed state-space and noise-correlated ARMAX models are analyzed. Because of equivalence between state-space LQG and ARMAX MV control laws, there is a full recovery in the delay-free $(d = 1)$ case. Obviously, occurrence of any time-delay and noise correlation in any configuration deteriorates the recovery in any discrete-time system.

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