Averaged controllability of heat equation in unbounded domains with random geometry and location of controls: The Green's function approach

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The constrained averaged controllability of linear one-dimensional heat equation defined on $\mathbb{R}$ and $\mathbb{R}_+$ is studied. The control is carried out by means of the time-dependent intensity of a heat source located at an uncertain interval of the corresponding domain, the end-points of which are considered as uniformly distributed random variables. Employing the Green's function approach, it is shown that the heat equation is not constrained averaged controllable neither in $\mathbb{R}$ nor in $\mathbb{R}^+$. Sufficient conditions on initial and terminal data for the averaged exact and approximate controllabilities are obtained. However, constrained averaged controllability of the heat equation is established in the case of point heat source, the location of which is considered as a uniformly distributed random variable.

Moreover, it is obtained that the lack of averaged controllability occurs for random variables with arbitrary symmetric density function.

Key words: lack of controllability, constrained controllability, heuristic method, averaged dynamics, uniformly distributed random variable

1. Introduction

The exact controllability of the heat equation defined on $\mathbb{R}_+$ and in $\mathbb{R}^3_+$ has been studied by Micu and Zuazua, respectively, in [1, 2] and [3] concluding the lack of its L^2-boundary null controllability. This result has been confirmed

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in [4] as a limiting case when the control source approaches the boundary of $\mathbb{R}_+$, straightforwardly using the Green's function approach, which turns to be very efficient in controllability analysis [5]. Moreover, as long as the control source is far from the boundary, i.e., distributed controllability is studied, necessary and sufficient conditions on the initial and terminal data are obtained for exact controllability of the heat equation on $\mathbb{R}_+$. Thus, the location of the control strongly affects the controllability property of the heat equation. This motivates to study the controllability of the heat equation on $\mathbb{R}_+$ and $\mathbb{R}$ when the location of the control is not fixed.

Let the heat equation defined on $\mathbb{R}$ or $\mathbb{R}_+$ be controlled by the intensity u of a heat source distributed at an uncertain bounded interval $[x_0, x_1]$. It is assumed that x_0 and x_1 are normally distributed independent random variables such that $|x_1 - x_0| \leqslant l$ for a given finite $l > 0$. The probability density function is then given by

$$\rho(x_0, x_1) = \frac{1}{\mu(\Omega_l)} \chi_{\Omega_l}(x_0, x_1),$$

where $\mu(\Omega_l)$ is the measure, χ_{Ω_l} is the indicator function of

$$\Omega_l = \{x_0, x_1 \in \Omega, \ |x_1 - x_0| \leqslant l\}.$$

Hereinafter, $\Omega = \mathbb{R}^+$ or $\Omega = \mathbb{R}$. Note also that when $\Omega = \mathbb{R}^+$, $\Omega_l = [0, l]^2$, $\mu(\Omega_l) = l^2$, and when $\Omega = \mathbb{R}$, $\Omega_l = [-l, l]^2$, $\mu(\Omega_l) = 4l^2$.

Given any initial, Θ_0, and desired, Θ_T, temperature distributions, finite control time T, and finite $l > 0$ such that $|x_1 - x_0| \leqslant l$, determine admissible functions $u \in \mathcal{U}$, such that at $t = T$

$$\mathcal{R}_T(u) = \|\Theta(x, T; u, \Theta_0, x_0, x_1) - \Theta_T(x)\| = 0,$$

where Θ is the temperature distribution in $\Omega \times [0, T]$, $\|\cdot\|$ is the norm of the space of the terminal states assumed to be $L^2(\Omega)$. However, this makes the control to be dependent on random variables x_0 and x_1. In order to avoid such situations, the concept of averaged control has been recently introduced by the prominent mathematician Enrique Zuazua in [6] as a property of parameter-dependent systems. Accoring to Zuazua, instead of $\mathcal{R}_T$ above, the following residue must be considered:

$$\mathcal{R}_T^{av}(u) = \left\|\mathbb{M}_T^{\Omega_l}[\Theta] - \Theta_T(x)\right\|_{L^2(\Omega)}, \tag{1}$$

where

$$\mathbb{M}_T^{\Omega_l}[\Theta] = \int_{\Omega_l} \Theta(x, T; u, \Theta_0, x_0, x_1) \, \mathrm{d}\mathbb{P}(x_0, x_1)$$

is the averaged temperature distribution or the mathematical expectation of Θ.

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Averaged controllability of various systems has been studied by Zuazua and colleagues [7–10]. In [9] a handful of open problems related to averaged controllability of parameter-dependent systems described by evolution partial differential equations is presented. Among very recent results, note the analysis of optimal averaged controllability of the wave equation with an unknown wave velocity [11] and the similar study of the averaged controllability of the N-dimensional heat equation defined on a bounded domain [12]. Besides an unknown parameter in the state equation, there is a missing boundary condition in both studies.

Most of the literature in this direction so far has been concerned with unconstrained controllability, and little is known for the non-trivial case when the control is restricted to take on values in a preassigned subset of the control space. In this paper, however, constrained averaged controllability is considered following to [13–16]. More specifically, the set of admissible controls is defined as $\mathcal{U} = \left\{ u \in L^2[0, T], \ |u| \leqslant \epsilon, \ \text{supp}(u) \subseteq [0, T] \right\}$.

In terms of the residue (1) two main definitions of averaged controllability are distinguished below.

Definition 1 *The state Θ is called constrained exactly averaged controllable if for any given Θ_0 and Θ_T, control time T, there exits an admissible control $u \in \mathcal{U}$ such that $\mathcal{R}_T^{av}(u) = 0$.*

Definition 2 *The state Θ is called constrained approximately averaged controllable if for any given Θ_0 and Θ_T, control time T, there exits an admissible control $u \in \mathcal{U}$ such that $\mathcal{R}_T^{av}(u) \leqslant \varepsilon$ for a required accuracy $\varepsilon > 0$.*

Hereinafter, shorter terms exactly averaged controllable and approximately averaged controllable will be used for convenience.

The aim of the present paper is to study exact and approximate controllability of the heat equation on $\mathbb{R}$ and $\mathbb{R}_+$ when the control heat source has a random geometry and location, i.e., x_0 and x_1 are random. At this, for the sake of simplicity, it is assumed that these are uniformly distributed independent random variables. It is then shown that the heat equation is not averaged controllable in the sense of Definitions 1 and 2 (Section 3). Moreover, it is also shown that the lack of controllability occurs as soon as the probability density function is symmetric with respect to x_0 and x_1 (Section 5). Nonetheless, when the heat source has a point distribution, exact and averaged controllability can still be established (Section 4).

2. Governing equation and its Green's function solution

In dimensionless variables and quantities, the heat diffusion in the rod in time is described by the system

$$
\begin{cases}
\dfrac{\partial \Theta}{\partial t} = \alpha \dfrac{\partial^2 \Theta}{\partial x^2} + u(t)\chi_{\Omega_l}(x), & x \in \Omega, \ t \in \mathbb{R}^+, \\
\Theta = \Theta_0, & x \in \Omega, \ t = 0,
\end{cases}
\tag{2}
$$

in which $\alpha > 0$ represents physical characteristics of the rod and is assumed to be given. When $\Omega = \mathbb{R}^+$, $\Theta \equiv 0$ for $x = 0$ and $t > 0$.

The problem is to describe the set $\mathcal{U}_{res}^{ex} = \{u \in \mathcal{U}, \ \mathcal{R}_T^{av}(u) = 0\}$ of exactly resolving average controls. In case when $\mathcal{U}_{res}^{ex} = \varnothing$ corresponding to the lack of exact average controllability, the description of the set $\mathcal{U}_{res}^{ap} = \{u \in \mathcal{U}, \ \mathcal{R}_T^{av}(u) \leqslant \varepsilon\}$ of approximately resolving average controls is required. If $\mathcal{U}_{res}^{ap} = \varnothing$, then a lack of approximate averaged controllability occurs. Evidently, $\mathcal{U}_{res}^{ap} = \varnothing$ implies $\mathcal{U}_{res}^{ex} = \varnothing$.

Remark 1 *When $x_0 \to x_1$, $|x_1 - x_0|^{-1} \chi_{[x_0,x_1]}(x) \to \delta(x - x_0)$ in the sense of distributions, where δ is Dirac's delta function. Exact and approximate controllability of heat equation with point heat source is studied in [4].*

2.1. Green's function solution

Involving Green's function solution of (2), the dependence $\mathcal{R}_T^{av} = \mathcal{R}_T^{av}(u)$ is made explicit, simplifying the averaged controllability analysis. Following to [18],

$$
\Theta(x,t;x_0,x_1) = \int_\Omega G(x,\xi,t)\,\Theta_0(\xi)\,\mathrm{d}\xi + \int_0^t \widehat{G}(x,t-\tau;x_0,x_1)\,u(\tau)\,\mathrm{d}\tau, \tag{3}
$$

where

$$
G(x,\xi,t) = \begin{cases} \varphi(x-\xi,t), & \Omega = \mathbb{R}, \\ \varphi(x-\xi,t) - \varphi(x+\xi,t), & \Omega = \mathbb{R}^+, \end{cases}
$$

$$
\widehat{G}(x,t;x_0,x_1) = \int_\Omega G(x,\xi,t)\,\chi_{\Omega_l}(\xi)\,\mathrm{d}\xi =
$$

$$
= \begin{cases} \psi(x-x_0,t) - \psi(x-x_1,t), & \Omega = \mathbb{R}, \\ \psi(x-x_0,t) + \psi(x+x_0,t) - \\ \quad -\psi(x-x_1,t) - \psi(x+x_1,t), & \Omega = \mathbb{R}^+, \end{cases}
$$

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$$\varphi(x, t) = \frac{1}{\sqrt{4\pi\alpha t}} \exp\left[-\frac{x^2}{4\alpha t}\right], \qquad \psi(x, t) = \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

and erf is the Gauss error function.

3. Lack of averaged controllability

Making use of the representation (3) evaluated at $t = T$, the mathematical expectation can be transformed into

$$
\mathbb{M}_T^{\Omega_l}[\Theta] = \int_{\Omega_l}\left[\int_{\Omega} G(x, \xi, T)\,\Theta_0(\xi)\mathrm{d}\xi +\right.
$$

$$
\left. + \int_0^T \widehat{G}(x, T - \tau; x_0, x_1)\,u(\tau)\mathrm{d}\tau\right]\rho(x_0, x_1)\mathrm{d}\Omega_l = \tag{4}
$$

$$
= \int_{\Omega} G(x, \xi, T)\,\Theta_0(\xi)\mathrm{d}\xi + \frac{1}{\mu(\Omega_l)}\int_0^T \widetilde{G}(x, T - \tau)\,u(\tau)\mathrm{d}\tau,
$$

where

$$
\widetilde{G}(x, t) = \int_{\Omega_l} \widehat{G}(x, t; x_0, x_1)\,\mathrm{d}\Omega_l.
$$

The first term in the right hand side of (4) does not depend on x_0 or x_1, so it can be combined with Θ_T in (1). Computing the integral in the second term (see Appendix A), it is obtained that $\widetilde{G} \equiv 0$ in $\Omega \times \mathbb{R}_+$ in both cases of $\Omega = \mathbb{R}$ and $\Omega = \mathbb{R}^+$. Then, the residue (1) becomes independent of u:

$$
\mathcal{R}_T^{\mathrm{av}}(u) = \left\|\int_{\Omega} G(x, \xi, T)\,\Theta_0(\xi)\mathrm{d}\xi - \Theta_T(x)\right\|_{\Theta_T}. \tag{5}
$$

Remark 2 *Evidently, when, e.g., $\Theta_0 = 0$ and $\Theta_T \neq 0$, then $\mathcal{R}_T^{\mathrm{av}} \neq 0$ for any $u \in \mathcal{U}$. In other words, $\mathcal{U}_{res}^{ex} = \varnothing$.*

Also, in general, for arbitrary Θ_0 and Θ_T, $\mathcal{R}_T^{\mathrm{av}} \leqslant \varepsilon$ may not be satisfied for sufficiently small accuracy ε. In other words, in general, $\mathcal{U}_{res}^{ap} = \varnothing$.

In general, the following assertion holds.

Theorem 1 *The system (2) is not averaged controllable by any $u \in \mathcal{U}$ neither in $\mathbb{R}$ nor $\mathbb{R}_+$.*

On the other hand, the following assertions hold.

Corollary 1 *If for prescribed Θ_0, Θ_T and T, the equality*

$$\Theta_T(x) = \int_\Omega G(x, \xi, T)\, \Theta_0(\xi)\, d\xi \tag{6}$$

holds, then the linear heat equation is exactly averaged controllable for every $u \in \mathcal{U}$.

Corollary 2 *If for prescribed Θ_0, Θ_T and T, the inequality*

$$\left\| \int_\Omega G(x, \xi, T)\, \Theta_0(\xi)\, d\xi - \Theta_T(x) \right\|_{L^2(\Omega)} < \varepsilon$$

holds, then the linear heat equation is approximately averaged controllable for every $u \in \mathcal{U}$.

Remark 3 *It is possible to keep Θ_0 arbitrary. Then, the set of reachable terminal states will be defined as*

$$\mathcal{T} = \left\{ \Theta_T \in L^2(\Omega),\ (6) \right\}.$$

4. Limiting case: point source

Recalling Remark 1, let $x_0 \to x_1$ corresponding to a point heat source. In that case, (3) is reduced to

$$\Theta(x, t; x_0) = \int_0^t G(x, x_0, t - \tau)\, u(\tau)\, d\tau + \int_\Omega G(x, \xi, t)\, \Theta_0(\xi)\, d\xi.$$

The averaged residue in this case will read as

$$\mathcal{R}_T^{\text{av}}(u) = \left\| \frac{1}{\mu(\Omega_l)} \int_0^T \widetilde{G}(x, T - \tau)\, u(\tau)\, d\tau + \int_\Omega G(x, \xi, T)\, \Theta_0(\xi)\, d\xi - \Theta_T(x) \right\|_{\Theta_T},$$

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where $\Omega_l = \{x_0 \in \Omega, \ |x_0| \leqslant l\}$, $\mu(\Omega_l)$ is the length of Ω_l, i.e., $\mu(\Omega_l) = 2l$ when $\Omega = \mathbb{R}$, and $\mu(\Omega_l) = l$ when $\Omega = \mathbb{R}^+$. In this case, however (see Appendix A),

$$
\widetilde{G}(x,t) = \int_{\Omega_l} G(x, x_0, t) \, dx_0 =
$$

$$
= \begin{cases} \psi(l - x, t) - \psi(l - x, t), & \Omega = \mathbb{R}, \\ 2\psi(x, t) - \psi(l - x, t) - \psi(l + x, t), & \Omega = \mathbb{R}^+. \end{cases}
$$

Then, the exact controllability of the system is achieved by $u \in \mathcal{U}$ constrained by

$$
\int_0^T \widetilde{G}(x, T - \tau) u(\tau) d\tau = M(x), \quad x \in \Omega, \tag{7}
$$

where

$$
M(x) = \mu(\Omega_l) \left[\Theta_T(x) - \int_\Omega G(x, \xi, T) \Theta_0(\xi) d\xi \right].
$$

Repeating the steps of the Green's function approach, appropriate constraints on $u \in \mathcal{U}$ will be derived for exact and approximate averaged controllability. For details of the steps, see [4, 5, 19]. Involving the heuristic method [17], the corresponding sets of resolving controls can be described.

5. Remarks on other types of random variables

The lack of averaged controllability obtained in Theorem 1 was established when x_0 and x_1 are uniformly distributed independent random variables. However, it is straightforwardly proved that when x_0 and x_1 are normally distributed independent random variables, then (2) is not averaged controllable. Indeed, in that case,

$$
\rho(x_0, x_1) = \frac{1}{2\pi} \exp \left[-\frac{x_0^2 + x_1^2}{2} \right]. \tag{8}
$$

Therefore, instead of (4), the averaged state is described by the expectation

$$
\mathbb{M}_T^{\Omega_l}[\Theta] = \frac{1}{2\pi} \int\limits_{\Omega_l} \left[\int\limits_{\Omega} G(x,\xi,T)\Theta_0(\xi)\,d\xi + \right.
$$

$$
\left. + \int\limits_0^T \widehat{G}(x,T-\tau;x_0,x_1)u(\tau)\,d\tau \right] \exp\left[-\frac{x_0^2 + x_1^2}{2}\right] d\Omega_l = \tag{9}
$$

$$
= \frac{1}{2\pi}\mu(\Omega_l) \int\limits_{\Omega} G(x,\xi,T)\Theta_0(\xi)\,d\xi + \frac{1}{2\pi} \int\limits_0^T \widetilde{G}_n(x,T-\tau)u(\tau)\,d\tau,
$$

where

$$
\widetilde{G}_n(x,t) = \int\limits_{\Omega_l} \widehat{G}(x,t;x_0,x_1) \exp\left[-\frac{x_0^2 + x_1^2}{2}\right] d\Omega_l.
$$

Evaluating the last integral, it is obtained that $\widetilde{G}_n \equiv 0$ in $\Omega \times \mathbb{R}_+$ in both cases of $\Omega = \mathbb{R}$ and $\Omega = \mathbb{R}_+$ (see Appendix). This results in independence of the averaged residue $\mathcal{R}_T^{\mathrm{av}}$ of u. Thus, Theorem 1 holds.

Generalizing, it turns out that as soon as the density function of the distribution of x_0 and x_1 is symmetric with respect to its argument, then system (2) is not averaged controllable neither on $\mathbb{R}$ nor on $\mathbb{R}_+$. In other words,

Theorem 2 *If $\rho(x_0,x_1) = \rho(x_0,x_1)$, then the statement of Theorem 1 holds true.*

Despite the difficulty of explicit determination of integrals in the limiting case of point source considered in Section 4 for complicated forms of ρ (e.g., (8)), it is still possible to derive controllability conditions in the form of (7).

6. Conclusion

Linear one-dimensional heat equation defined in $\mathbb{R}$ and $\mathbb{R}_+$ is governed by the time-dependent intensity of a heat source, the location and geometry of which is assumed to be bounded, uniformly distributed random variables. Involving the Green's function approach, the mathematical expectation or the averaged dynamics is represented explicitly. It is established that the heat equation lacks to be constrained averaged controllable in finite time unless constrained initial and terminal states are considered. On the other hand, it is shown that constrained

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averaged controllability, in principle, can be achieved in the case of heat source with point distribution.

In addition, it is proved that when the probability density function is symmetric with respect to the random variables, then the heat equation is not averaged controllable.

Appendix. Evaluation of some integrals

The case of uniformly distributed random variables

Begin with the evaluation of $\widetilde{G}$. First, consider the case when $\Omega = \mathbb{R}$. Then, $\Omega_l = [-l, l] \times [-l, l]$,

$$
\widetilde{G}(x, t) = \int_{-l}^{l} \int_{-l}^{l} \widehat{G}(x, t; x_0, x_1) \, dx_0 dx_1 =
$$

$$
= \int_{-l}^{l} \int_{-l}^{l} [\psi(x - x_0, t) - \psi(x - x_1, t)] \, dx_0 dx_1 =
$$

$$
= l \left[\int_{-l}^{l} \psi(x - x_0, t) dx_0 - \int_{-l}^{l} \psi(x - x_1, t) dx_1 \right].
$$

Evidently

$$
\int_{-l}^{l} \psi(x - x_0, t) dx_0 - \int_{-l}^{l} \psi(x - x_1, t) dx_1 \equiv 0
$$

in $\mathbb{R} \times [0, T]$. In the same way it is proved that $\widetilde{G} \equiv 0$ for $x \in \mathbb{R}^+$, $t \in [0, T]$.

Now, compute the integral

$$
\widetilde{G}(x, t) = \int_{\Omega_l} G(x, x_0, t) dx_0.
$$

For the sake of simplicity, consider the case when $\Omega = \mathbb{R}$. Then, $\Omega_l = [-l, l]$ and $\mu(\Omega_l) = 2l$. Therefore, taking into account that

$$
\int \varphi(x - x_0, t) dx_0 = -\psi(x - x_0, t),
$$

it is immediately derived that

$$
\widetilde{G}(x, t) = \psi(l + x, t) - \psi(l - x, t).
$$

The case of normally distributed random variables

Now, evaluate the integrals arising in the case when x_0 and x_1 are standard normally distributed random variables. Assume that $\Omega = \mathbb{R}$. Then,

$$\widetilde{G}_n(x, t) = \int_{-l}^{l} \int_{-l}^{l} [\psi(x - x_0, t) - \psi(x - x_1, t)] \exp\left[-\frac{x_0^2 + x_1^2}{2}\right] dx_0 dx_1 =$$

$$= \int_{-l}^{l} \psi(x - x_0, t) \left(\int_{-l}^{l} \exp\left[-\frac{x_0^2 + x_1^2}{2}\right] dx_1\right) dx_0 -$$

$$- \int_{-l}^{l} \psi(x - x_1, t) \left(\int_{-l}^{l} \exp\left[-\frac{x_0^2 + x_1^2}{2}\right] dx_0\right) dx_1 =$$

$$= \sqrt{2\pi}\, \mathrm{erf}\left(\frac{l}{\sqrt{2}}\right)\left[\int_{-l}^{l} \psi(x - x_0, t) \exp\left[-\frac{x_0^2}{2}\right] dx_0 -\right.$$

$$\left.- \int_{-l}^{l} \psi(x - x_1, t) \exp\left[-\frac{x_1^2}{2}\right] dx_1\right] \equiv 0.$$

The same reasoning will end up with the conclusion that when $\Omega = \mathbb{R}_+$,

$$\widetilde{G}_n(x, t) = \int_{-l}^{l} \int_{-l}^{l} [\psi(x - x_0, t) - \psi(x - x_1, t)+$$

$$+ \psi(x + x_0, t) - \psi(x + x_1, t)] \exp\left[-\frac{x_0^2 + x_1^2}{2}\right] dx_0 dx_1 \equiv 0.$$

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