BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 68, No. 6, 2020
DOI: 10.24425/bpasts.2020.135376

The pointwise completeness and the pointwise degeneracy of linear discrete-time different fractional order systems

T. KACZOREK and Ł. SAJEWSKI*

Bialystok University of Technology, Faculty of Electrical Engineering, ul. Wiejska 45D, 15-351 Białystok, Poland

Abstract. Necessary and sufficient conditions for the pointwise completeness and the pointwise degeneracy of linear discrete-time different fractional order systems are established. It is shown that if the fractional system is pointwise complete in one step ($q = 1$), then it is also pointwise complete for $q = 2, 3\ldots$.

Key words: discrete-time, fractional, different order, pointwise completeness, pointwise degeneracy, positive.

1. Introduction

A dynamical system described by homogenous equation is called pointwise complete if every final state of the system can be reached by a suitable choice of its initial state. A system which is not pointwise complete is called pointwise degenerated. Pointwise completeness and pointwise degeneracy together with controllability and observability, apart from stability, constitute basic properties of dynamical systems which must be taken into consideration during the control synthesis problems.

The pointwise completeness and pointwise degeneracy of continuous-time linear systems with delays have been investigated in [1–3], the pointwise completeness of discrete-time linear systems with delays in [4, 5] and of fractional linear systems in [6–9]. The pointwise completeness and the pointwise degeneracy of standard and positive hybrid systems described by the general model have been analyzed in [10] and of positive linear systems with state-feedbacks in [11]. Some new results in fractional systems have been given in [3, 12–16, 17]. The Drazin inverse of matrices has been applied to the analysis of pointwise completeness and of pointwise degeneracy of the descriptor continuous-time and discrete-time linear systems in [18, 19] and for fractional standard descriptor continuous-time linear systems in [20] and discrete-time linear systems in [7].

In this paper pointwise completeness and pointwise degeneracy of discrete-time different fractional order linear systems will be considered.

The paper is organized as follows. In Sec. 2 different fractional order discrete-time linear systems and their positivity are presented. The pointwise completeness of fractional discrete-time linear systems is investigated in Sec. 3 and the pointwise degeneracy in Sec. 4. Concluding remarks are given in Sec. 5. The considerations are illustrated by numerical examples of fractional different order discrete-time linear systems.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n – the $n\times n$ identity matrix.

*e-mail: l.sajewski@pb.edu.pl

Manuscript submitted 2019-10-10, revised 2020-07-07, initially accepted for publication 2020-07-29, published in December 2020

2. Fractional discrete-time linear systems with different orders and its positivity

Consider the autonomous fractional discrete-time linear system with two different fractional orders α and β of the form:

$$\begin{aligned}\Delta^\alpha x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k),\\ \Delta^\beta x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k),\end{aligned} \tag{1}$$

where $k \in Z_+ = 0,1,\ldots$; $x_1(k) \in \Re^{n_1}$, $x_2(k) \in \Re^{n_2}$ and $A_{ij} \in \Re^{n_i\times n_j}$; $i, j = 1,2$, $n = n_1 + n_2$.

The fractional difference of α (β) order is defined by [7]:

$$\begin{aligned}&\Delta^\alpha x(k) = \sum_{j=0}^{k}(-1)^j\binom{\alpha}{j}x(k-j) = \sum_{j=0}^{k}c_\alpha(j)x(k-j),\\ &c_\alpha(j) = (-1)^j\binom{\alpha}{j} = (-1)^j\frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!},\\ &c_\alpha(0) = 1, \qquad j = 1,2,\ldots\end{aligned} \tag{2}$$

Using (2) we can write the equation (1) in the matrix form:

$$\begin{aligned}\begin{bmatrix}x_1(k+1)\\ x_2(k+1)\end{bmatrix} &= \begin{bmatrix}A_{1\alpha} & A_{12}\\ A_{21} & A_{2\beta}\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}\\ &-\sum_{j=2}^{k+1}\begin{bmatrix}c_\alpha(j)I_{n_1} & 0\\ 0 & c_\beta(j)I_{n_2}\end{bmatrix}\begin{bmatrix}x_1(k-j+1)\\ x_2(k-j+1)\end{bmatrix},\end{aligned} \tag{3}$$

where $A_{1\alpha} = A_{11} + I_{n_1}\alpha$, $A_{2\beta} = A_{22} + I_{n_2}\beta$.

Theorem 1. The solution to the fractional system described by Eqs. (1) with initial conditions $x_1(0) = x_{10}$, $x_2(0) = x_{20}$ is given by:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \Phi_k \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}, \qquad k \in Z_+, \tag{4}$$

where Φ_k is defined by:

$$\Phi_k = \begin{cases} I_{n_1+n_2} & \text{for } k = 0 \\ A_{\alpha\beta}\Phi_{k-1} - D_1\Phi_{k-2} - \ldots - D_{k-1}\Phi_0 & \text{for } k = 1,2,\ldots,i, \\ A_{\alpha\beta}\Phi_{k-1} - D_1\Phi_{k-2} - \ldots - D_i\Phi_{k-i-1} & \text{for } k = i+1, i+2, \ldots \end{cases} \tag{5a}$$

and

$$A_{\alpha\beta} = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix}, \quad D_k = \begin{bmatrix} c_\alpha(k+1)I_{n_1} & 0 \\ 0 & c_\beta(k+1)I_{n_2} \end{bmatrix}. \tag{5b}$$

The proof is given in [8].

Definition 1. The fractional system (1) is called positive if and only if $x_1(k) \in \Re_+^{n_1}$, $x_2(k) \in \Re_+^{n_2}$, $k \in Z_+$ for any initial conditions $x_1(0) = x_{10} \in \Re_+^{n_1}$, $x_2(0) = x_{20} \in \Re_+^{n_2}$.

Theorem 2. The fractional discrete-time linear system (1) with $0 < \alpha < 1$, $0 < \beta < 1$ is positive if and only if:

$$A_{\alpha\beta} = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \in \Re_+^{n\times n}. \tag{6}$$

The proof is given in [8].

Theorem 3. If the fractional discrete-time linear system (1) is positive then the matrix Φ_k defined by (5a) has nonnegative entries, i.e.

$$\Phi_k \in \Re_+^{n\times n} \quad \text{for } k \in Z_+. \tag{7}$$

Proof. If $0 < \alpha < 1$ and $0 < \beta < 1$ then from (2) we have $c_\alpha(j) < 0$ ($c_\beta(j) < 0$) for $j \in Z_+$ and from (5b) we obtain $-D_k \in \Re_+^{n\times n}$ for $k \in Z_+$. In this case from (5a) we obtain (7).□

Example 1. Consider the autonomous fractional different order system (1) with $\alpha = 0.6$, $\beta = 0.8$ and:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -0.5 & 0.2 & 0.1 & 0 \\ 0.1 & -0.4 & 0 & 0.2 \\ 0 & 0.1 & -0.6 & 0.2 \\ 0.2 & 0 & 0.1 & -0.7 \end{bmatrix}, \tag{8}$$

$n_1 = n_2 = 2$.

Using (5b) and (8) we obtain:

$$A_{\alpha\beta} = \begin{bmatrix} A_{11} + I_{n_1}\alpha & A_{12} \\ A_{21} & A_{22} + I_{n_2}\beta \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0 \\ 0.1 & 0.2 & 0 & 0.2 \\ 0 & 0.1 & 0.2 & 0.2 \\ 0.2 & 0 & 0.1 & 0.1 \end{bmatrix} \in \Re_+^{4\times 4}. \tag{9}$$

Using (5a), (5b) and (9) we obtain:

$$\begin{aligned} &\Phi_0 = I_4, \\ &\Phi_1 = A_{\alpha\beta} = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0 \\ 0.1 & 0.2 & 0 & 0.2 \\ 0 & 0.1 & 0.2 & 0.2 \\ 0.2 & 0 & 0.1 & 0.1 \end{bmatrix} \in \Re_+^{4\times 4}, \\ &\Phi_2 = A_{\alpha\beta}^2 - D_1\Phi_0 \\ &\quad = \begin{bmatrix} 0.15 & 0.07 & 0.03 & 0.06 \\ 0.07 & 0.18 & 0.03 & 0.06 \\ 0.05 & 0.04 & 0.14 & 0.08 \\ 0.04 & 0.05 & 0.05 & 0.11 \end{bmatrix} \in \Re_+^{4\times 4}, \\ &\Phi_3 = A_{\alpha\beta}\Phi_2 - D_1\Phi_1 - D_2\Phi_0 \\ &\quad = \begin{bmatrix} 0.102 & 0.071 & 0.035 & 0.026 \\ 0.049 & 0.133 & 0.019 & 0.064 \\ 0.025 & 0.044 & 0.089 & 0.06 \\ 0.055 & 0.023 & 0.033 & 0.071 \end{bmatrix} \in \Re_+^{4\times 4}. \end{aligned} \tag{10}$$

Therefore, by Theorem 2 the fractional system (1) with (8) is positive and by Theorem 3 the matrix $\Phi_k \in \Re_+^{4\times 4}$ for $k = 0, 1, 2, \ldots$ This confirms Theorem 3.

Theorem 4. The fractional discrete-time linear system (1) with $0 < \alpha < 1$, $0 < \beta < 1$ is positive and asymptotically stable if and only if there exists strictly positive vector $\lambda > 0$ such that:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \lambda < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{11}$$

Proof. Similar to the proof of the system:

$$\Delta^\alpha x(k+1) = Ax(k), \qquad 0 < \alpha < 1, \tag{12}$$

which is positive and asymptotically stable if $\exists\, \lambda > 0$ such that $A\lambda < 0$ [8]. □

3. Pointwise completeness of discrete-time different fractional order linear systems

In this section necessary and sufficient condition for the pointwise completeness of discrete-time different fractional order linear systems will be established.

Definition 2. The discrete-time different fractional order linear system (1) is called pointwise complete for $k = q$ if for final state $x_f \in \Re^n$ there exists an initial condition $x_0 \in \Re^n$ such that $x_q = x_f$.

Theorem 5. The discrete-time different fractional order linear system (1) is pointwise complete for any $k = q$ and every $x_f \in \Re^n$ if and only if:

$$\det \Phi_q \neq 0. \tag{13}$$

Proof. From (4) for $k = q$ it follows that:

$$x_0 = \Phi_q^{-1} x_f \tag{14}$$

if and only if the condition (13) is satisfied. □

Theorem 6. If the discrete-time different fractional order linear system (1) is pointwise complete for $q = 1$ then it is also pointwise complete for $q = 2, 3, \ldots$.

Proof. By Theorem 5 the fractional system (1) is pointwise complete for $q = 1$ if and only if $\det \Phi_1 = \det A_{\alpha\beta} \neq 0$. Taking into account that $\Phi_2 = A_{\alpha\beta}^2 - D_1\Phi_0$ and that: 1) the eigenvalues of the matrix $A_{\alpha\beta}^2$ are $\lambda_1^2, \ldots, \lambda_n^2$ if the eigenvalues of $A_{\alpha\beta}$ are $\lambda_1, \ldots, \lambda_n$; 2) the eigenvalues of the matrix Φ_2 are $\lambda_1^2 + d_1, \ldots, \lambda_n^2 + d_n$ for $D_1 = \mathrm{diag}\,[d_1, \ldots, d_n]$, $d_i > 0$, $i = 1, \ldots, n$ therefore, the matrix Φ_2 is nonsingular since $\det \Phi_2 = (\lambda_1^2 + d_1)\ldots(\lambda_n^2 + d_n) \neq 0$. Continuing this procedure we may show that $\det \Phi_i \neq 0$ for $i = 3, 4, \ldots$. □

Example 2. (Continuation of Example 1) Consider the autonomous fractional different order system (1) with $\alpha = 0.6$, $\beta = 0.8$ and the matrix (8).

Using (10) we obtain:

$$\begin{aligned} \det \Phi_0 &= \det I_4 = 1 \neq 0, \\ \det \Phi_1 &= \det A_{\alpha\beta} = 0.0023 \neq 0, \\ \det \Phi_2 &= 1.9893 \cdot 10^{-4} \neq 0, \\ \det \Phi_3 &= 4.6493 \cdot 10^{-5} \neq 0. \end{aligned} \tag{15}$$

Therefore, by Theorem 5 the different fractional order system (1) with (10) is pointwise complete for $q = 1, 2, 3$. The result (15) confirms Theorem 6.

Conclusion 1. The discrete-time different fractional order linear system (1) is pointwise complete if it is positive.

4. Pointwise degeneracy of discrete-time different fractional order linear systems

In this section necessary and sufficient conditions for the pointwise degeneracy of discrete-time different fractional order linear systems will be established.

Definition 3. The discrete-time different fractional order linear system (1) is called pointwise degenerated in the direction $v \in \Re^n$ for $k = q$ if there exists a nonzero vector v such that for all initial conditions $x_0 \in \Re^n$ the solution of the system (1) for $k = q$ satisfies the condition:

$$v^T x_q = 0. \tag{16}$$

Theorem 7. The discrete-time different fractional order linear systems (1) is pointwise degenerated in the direction $v \in \Re^n$ for $k = q$ if and only if:

$$\det \Phi_q = 0. \tag{17}$$

Proof. From (16) and (4) for $k = q$ we have:

$$v^T \Phi_q x_0 = 0. \tag{18}$$

Note that there exists nonzero vector $v \in \Re^n$ such that the condition (18) is satisfied for all $x_0 \in \Re^n$ if and only if the matrix Φ_q is singular. Therefore, the discrete-time different fractional order linear system (1) is pointwise degenerated in the direction $v \in \Re^n$ for $k = q$ if and only if the condition (17) is satisfied. □

From comparison of Theorem 5 and 7 we have the following important conclusion.

Conclusion 2. The discrete-time different fractional order linear system (1) is pointwise degenerated if and only if it is not pointwise complete.

Theorem 8. If the discrete-time different fractional order linear system (1) is not pointwise degenerated for $k = q = 1$, then it is also not pointwise degenerated for $q = 2, 3, \ldots$.

Proof. The proof follows from Theorem 7 and Conclusion 1. □

Example 3. (Continuation of Example 1 and 2) Consider the autonomous fractional different order system (1) with $\alpha = 0.6$, $\beta = 0.8$ and the matrix (8). From (15) it follows that $\det \Phi_i \neq 0$ for $i = 1, 2, 3$. Therefore, by Theorem 7 the system (1) with the matrix (8) is not pointwise degenerated for $q = 1, 2, 3$.

5. Concluding remarks

The pointwise completeness and of the pointwise degeneracy of the different fractional order discrete-time linear systems have been investigated. Necessary and sufficient conditions for the pointwise completeness and the pointwise degeneracy of the different fractional order discrete-time linear systems have been established (Theorems 5 and 7). It is shown that if the fractional different order discrete-time linear system is pointwise complete in one step ($q = 1$) then it is also pointwise complete for $q > 1$ (Theorems 6 and 8). The considerations have been illustrated by numerical examples of the different fractional order discrete-time linear systems. The considerations can be extended to the descriptor fractional different order discrete-time linear systems.

Acknowledgements. This work was supported by National Science Centre in Poland under work No. 2017/27/B/ST7/02443.

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