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Uniform exponential stabilization of distributed bilinear parabolic time delay systems with bounded feedback control

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In this paper we deal with the problem of uniform exponential stabilization for a class of distributed bilinear parabolic systems with time delay in a Hilbert space by means of a bounded feedback control. The uniform exponential stabilization problem of such a system reduces to stabilizing only its projection on a suitable finite dimensional subspace. Furthermore, the stabilizing feedback control depends only on the state projection on the finite dimensional subspace. An explicit decay rate estimate of the stabilized state is given provided that a non-standard weaker observability condition is satisfied. Illustrative examples for partial functional differential equations are displayed.

Key words: time delay, uniform exponential stabilization, bilinear parabolic systems, bounded feedback control

1. Introduction

The study of functional differential equations with time delay is motivated by the fact that the modelling of many evolution phenomena, arising in physics, biology or engineering sciences, often involves a time delay in the state variables. Many sources may induce delays such as the sensors or actuators response latency, the transmission time of information, or the computational time interval. The presence of time delay in the control or state variables, often encountered in engineering controlled systems, generally represents a source of instability which makes the use of an adequate feedback law necessary to remedy such disturbance.

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Stabilization of delayed systems has drawn the attention of many researchers. In the finite dimensional framework, time delay systems have been widely studied (see e.g. [3, 4, 11, 13] and the references therein). In [3], a robust H_∞ control has been designed to guarantee the robust stabilization of a cooperative driving system with time delay based on Lyapunov stability theory. An LMI approach was used in [4] to design adequate control laws leading to asymptotic stability of a linear retarded system. In [11], the exponential stabilization for a class of time varying delayed systems via impulsive control have been investigated. The problem of stabilization of a class of bilinear systems with delayed state and saturating actuators have been addressed in [13]. In the case of infinite dimensional systems, delayed systems have been considered for various purposes. In [8,9], the author considered optimal control problems involving parabolic delay partial differential equations. The authors in [12] studied the stabilization problem for a class of linear invariant systems governed by an abstract retarded functional differential equation with time delay in a Banach space. Asymptotic stabilization of semilinear distributed parameter control systems with time delay has been considered in [5].

In this paper, we deal with the question of feedback stabilization of distributed bilinear parabolic systems with time delay $r > 0$, described as follows:

$$\begin{cases} \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} = Az(t) + v(t)Bz(t-r), & t \geqslant 0, \\ z(t) = \varphi(t), & t \in [-r,\ 0]. \end{cases} \tag{1}$$

Here $z(t)$ denotes the state which lies in a Hilbert space H endowed with the inner product $\langle .,. \rangle$ and its corresponding norm $\|.\|$. Moreover, we assume that the linear operator $A\colon \mathcal{D}(A) \subset H \longrightarrow H$ (generally unbounded) generates a strongly continuous semigroup of contractions $S(t)$ on H (see [16, 21]), that is A is dissipative (i.e., $\langle A\psi,\ \psi \rangle \leqslant 0$, $\forall \psi \in \mathcal{D}(A)$). For $z \in C([-r, +\infty[, H)$ and $t \geqslant 0$, we define the function z^t as the element of the Banach space of continuous functions $C = C([-r,0], H)$ given by $z^t(\theta) = z(t+\theta)$ for $\theta \in [-r,0]$, defined from $[-r,\ 0]$ with values in H. The space C is equipped with the supremum norm $\|\psi\|_C = \sup\limits_{\theta\in[-r,0]} \|\psi(\theta)\|$ and $\varphi \in C$ denotes a given initial function. The operator B is a bounded linear operator defined from H into H, whereas $t \longmapsto v(t)$ is a scalar function which represents the control. In the sequel, we will present an appropriate decomposition of the state space H and system (1) using the spectral properties of the operator A and apply this approach to study the stabilization problem of system (1). This idea was previously used for linear systems in [1, 15, 17, 19]. Motivated by the above discussion, we will consider the strong stabilization problem for distributed bilinear systems with delay. In [7], it has been shown that if the spectrum $\sigma(A)$ of A can be decomposed into $\sigma_u(A) = \{\lambda : \mathrm{Re}(\lambda) \geqslant -\gamma\}$ and $\sigma_s(A) = \{\lambda : \mathrm{Re}(\lambda) < -\gamma\}$, for some $\gamma > 0$, such that $\sigma_u(A)$ can be separated from $\sigma_s(A)$ by a simple and closed curve Γ, then the state space H may be decomposed according to

$$H = H_u \oplus H_s\,, \tag{2}$$

where $H_u = P_u H$, $H_s = P_s H$, P_u is the projection operator given by

$$P_u = \frac{1}{2\pi i}\int_{\Gamma} (\lambda I - A)^{-1}\,\mathrm{d}\lambda, \tag{3}$$

and $P_s = I - P_u$. The projection operators P_u and P_s commute with A, and we have $A = A_u + A_s$ with $A_u = P_u A$ and $A_s = P_s A$. For all $z \in H$, we set $z_u = P_u z$ and $z_s = P_s z$. In addition, we suppose that the operator of control B satisfies:

$$B = B_u + B_s\,, \tag{4}$$

with $B_u = P_u B P_u$ and $B_s = P_s B P_s$ (see [20]). It has been shown in [6], under the spectrum growth assumption:

$$\lim_{t\to+\infty} \frac{\ln(\|S_s(t)\|)}{t} = \sup \mathrm{Re}(\sigma(A_s)), \tag{5}$$

where $S_s(t)$ denotes the semigroup generated by A_s in H_s, that system (1) is strongly stabilizable (*i.e.* there exists a control law that assures $\|z(t)\| \longrightarrow 0$ as $t \longrightarrow +\infty$) with the following rational decay rate estimate

$$\|z(t)\| = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \quad \text{as} \quad t \longrightarrow +\infty,$$

using the continuous feedback control

$$v(t) = -\rho\langle B_u z_u(t-r),\ z_u(t)\rangle, \quad \forall\, t \geqslant 0, \quad \rho > 0,$$

provided that the assumption:

$$\langle B_u S_u(t-r)\phi, S_u(t)\phi\rangle = 0, \quad \forall\, t \geqslant r \Rightarrow \phi = 0, \tag{6}$$

is satisfied. The main objective of this paper consists in using the decomposition method (2) to study the uniform exponential stabilization of system (1) with an explicit decay rate estimate of the stabilized state. For this purpose, we consider the case where (4) holds, and so the system (1) can be splitted into the following two subsystems:

$$\begin{cases} \dfrac{\mathrm{d}z_u(t)}{\mathrm{d}t} = A_u z_u(t) + v(t) B_u z_u(t-r), & t \geqslant 0, \\ z_u(t) = \varphi_u(t), & t \in [-r,\ 0], \end{cases} \tag{7}$$

and

$$\begin{cases} \dfrac{\mathrm{d}z_s(t)}{\mathrm{d}t} = A_s z_s(t) + v(t) B_s z_s(t-r), & t \geqslant 0, \\ z_s(t) = \varphi_s(t), & t \in [-r,\ 0], \end{cases} \tag{8}$$

in the spaces H_u and H_s respectively, with $\varphi(t) = \varphi_u(t) + \varphi_s(t)$, $\forall t \in [-r, 0]$, where $\varphi_u \in C_u := C([-r, 0], H_u)$ and $\varphi_s \in C_s := C([-r, 0], H_s)$.

The rest of this paper is organised as follows: In the next section, we will present some definitions which will be needed in our analysis. In Section 3, we will study the existence and uniqueness of the global mild solution of system (1). Furthermore, we will establish a useful estimate which will be used in the uniform exponential stabilization problem. Section 4 is devoted to the main results. More precisely, we will show the uniform exponential stabilization of system (1) with an explicit decay rate estimate of the stabilized state under a non-standard weaker observability condition. Some illustrating examples are presented in Section 5.

2. Some preliminaries

In this section, we recall some basic definitions concerning the asymptotic behaviour of system (1), which are useful for the forthcoming developments.

Definition 1 [21]. *Let $T > 0$. A function $z \in C([-r, T], H)$ is said to be a mild solution of system (1) if it satisfies:*

$$z(t) = \begin{cases} S(t)z(0) + \displaystyle\int_0^t S(t-\tau) F(\tau, z^\tau)\,\mathrm{d}\tau, & t \in [0, T], \\ \varphi(t), & t \in [-r,\ 0], \end{cases} \tag{9}$$

where the map $F\colon [0,\ T] \times C \longmapsto H$ is defined by $F(t, \phi) = v(t) B \phi(-r)$, $t \in [0,\ T]$.

Definition 2 *System (1) is said to be exponentially stabilizable, if there exists a feedback control $v(t) = g(z^t)$, $t \geqslant 0$, where g is a suitable real valued function defined on C such that*

1. *$z(t)$ is the unique mild solution on $[-r,\ +\infty)$ of system (1).*
2. *$\{0\}$ is a stable equilibrium of system (1).*
3. *There exist two constants $\sigma > 0$ and $M > 0$ such that*

$$\|z(t)\| \leqslant M \|\varphi\|_C e^{-\sigma t}, \quad \forall\, t \geqslant 0. \tag{10}$$

System (1) is uniformly exponentially stabilizable if the estimate (10) holds for some M and σ, which are independent of φ.

Remark 1

1. *The fact that P_u and P_s are two projection operators gives*

$$\max\{\|B_u\|, \|B_s\|\} \leqslant \|B\|. \tag{11}$$

2. *Note that if $S(t)$ is a semigroup of contractions, so $S_u(t)$ is. Indeed, since $S(t)$ is a semigroup of contractions, then $-A$ is maximal monotone (see [2]). Moreover, it is easily seen that $-A_u$ is also maximal monotone, so A_u generates a semigroup of contractions $S_u(t)$.*

3. *In this paper, the function g in Definition 2 lies in $L^\infty(C)$ so that the considered feedback control v belongs to $L^\infty(0,+\infty)$.*

3. Existence and uniqueness of the global mild solution and decay estimate

The first main result concerns the existence and uniqueness of the global mild solution of system (1). Moreover, we will provide a decay estimate which will be required later in our analysis.

Theorem 1 *Assume that A generates a semigroup of contractions $S(t)$ and $B \in \mathcal{L}(H)$ such that (4) and (5) hold. Then system (1) controlled by the feedback control law:*

$$v_r(t) = \begin{cases} -\rho \dfrac{\langle B_u z_u(t-r), z_u(t)\rangle}{\|z_u^t\|^2_{C_u}}, & t \geqslant 0,\ \rho > 0, \quad \|z_u^t\|_{C_u} \neq 0, \\ 0, & \qquad\qquad\qquad\qquad \|z_u^t\|_{C_u} = 0, \end{cases} \tag{12}$$

possesses a unique global mild solution $z \in C([-r,+\infty), H)$, and for any $t > 0$ the map $\varphi_u \longmapsto z_u^t$ is Lipschitz continuous from C_u to C_u. Moreover, for each $T > r$, we have

$$\int_r^T |\langle B_u S_u(\sigma - r) z_u(t), S_u(\sigma) z_u(t)\rangle| \mathrm{d}\sigma$$

$$= O\left[\left(\|z_u(t)\|^2 \int_t^{t+T} \frac{|\langle B_u z_u(\sigma - r), z_u(\sigma)\rangle|^2}{\|z_u^\sigma\|^2_{C_u}} \mathrm{d}\sigma\right)^{\frac{1}{2}}\right], \quad \forall\, t \geqslant 0. \tag{13}$$

Proof. To show that system (1) admits a unique mild solution by using the feedback control (12), it suffices to show that the two subsystems (7) and (8) do. Using the control expression (12), system (7) becomes:

$$\begin{cases} \dfrac{\mathrm{d}z_u(t)}{\mathrm{d}t} = A_u z_u(t) + G(z_u^t), & t \geqslant 0, \\ z_u(t) = \varphi_u(t), & t \in [-r,\ 0], \end{cases} \tag{14}$$

where the function $G\colon C_u \longrightarrow H_u$ is given by:

$$G(\phi) := \begin{cases} -\rho \dfrac{\langle B_u\phi(-r), \phi(0)\rangle}{\|\phi\|^2_{C_u}} B_u\phi(-r), & \|\phi\|_{C_u} \neq 0, \\ 0, & \|\phi\|_{C_u} = 0. \end{cases} \tag{15}$$

To prove that system (14) admits a unique mild solution we will prove that the function G is globally Lipschitz continuous. First, it can be easily seen that the function G satisfies the local Lipschitz condition if $y = 0$ or $z = 0$. Assume, for instance, that $0 < \|z\|_{C_u} \leqslant \|y\|_{C_u}$, we have

$$\begin{aligned}
&\|G(z) - G(y)\| \\
&\quad = \rho \frac{\left\| \|y\|^2_{C_u} \langle B_u z(-r), z(0)\rangle B_u z(-r) - \|z\|^2_{C_u} \langle B_u y(-r), y(0)\rangle B_u y(-r) \right\|}{\|y\|^2_{C_u} \|z\|^2_{C_u}} \\
&\quad \leqslant \rho \frac{\left\| \|y\|^2_{C_u} \langle B_u z(-r), z(0)\rangle B_u z(-r) - \|z\|^2_{C_u} \langle B_u z(-r), z(0)\rangle B_u z(-r) \right\|}{\|y\|^2_{C_u} \|z\|^2_{C_u}} \\
&\qquad + \rho \frac{\left\| \|z\|^2_{C_u} \langle B_u z(-r), z(0)\rangle B_u z(-r) - \|z\|^2_{C_u} \langle B_u y(-r), y(0)\rangle B_u y(-r) \right\|}{\|y\|^2_{C_u} \|z\|^2_{C_u}} \\
&\quad \leqslant \rho \|B\|^2 \frac{\|z\|^3_{C_u}}{\|y\|^2_{C_u} \|z\|^2_{C_u}} \left| \|y\|^2_{C_u} - \|z\|^2_{C_u} \right| \\
&\qquad + \rho \frac{\left\| \langle B_u z(-r), z(0)\rangle B_u z(-r) - \langle B_u y(-r), y(0)\rangle B_u y(-r) \right\|}{\|y\|^2_{C_u}} \\
&\quad \leqslant \rho \|B\|^2 \frac{\|z\|_{C_u}}{\|y\|^2_{C_u}} \left(\|y\|_{C_u} + \|z\|_{C_u} \right) \left| \|y\|_{C_u} - \|z\|_{C_u} \right| \\
&\qquad + \rho \frac{\left\| \langle B_u z(-r), z(0)\rangle B_u z(-r) - \langle B_u z(-r), z(0)\rangle B_u y(-r) \right\|}{\|y\|^2_{C_u}} \\
&\qquad + \rho \frac{\left\| \langle B_u z(-r), z(0)\rangle B_u y(-r) - \langle B_u y(-r), y(0)\rangle B_u y(-r) \right\|}{\|y\|^2_{C_u}}
\end{aligned}$$

$$\begin{aligned}
&\leqslant 2\rho\|B\|^2\|y-z\|_{C_u} + \rho\|B\|^2\frac{\|z\|^2_{C_u}}{\|y\|^2_{C_u}}\|y-z\|_{C_u}\\
&\quad+\rho\frac{\left\|\langle B_u z(-r), z(0)\rangle B_u y(-r) - \langle B_u y(-r), z(0)\rangle B_u y(-r)\right\|}{\|y\|^2_{C_u}}\\
&\quad+\rho\frac{\left\|\langle B_u y(-r), z(0)\rangle B_u y(-r) - \langle B_u y(-r), y(0)\rangle B_u y(-r)\right\|}{\|y\|^2_{C_u}}\\
&\leqslant K(\rho,\|B\|)\|z-y\|_{C_u},
\end{aligned}$$

where $K(\rho,\|B\|) := 5\rho\|B\|^2$. Therefore, the function G is globally Lipschitz continuous. Hence (14) possesses a unique global mild solution $z_u \in C([-r,+\infty), H_u)$ given by the variation of constants formula:

$$z_u(t) = \begin{cases} S_u(t)z_u(0) + \displaystyle\int_0^t S_u(t-\sigma)v_r(\sigma)B_u z_u(\sigma - r)\,\mathrm{d}\sigma, & t\in[0,+\infty),\\ \varphi_u(t), & t \in [-r,\ 0], \end{cases} \tag{16}$$

(see [21], Theorem 1.1, p. 37). Let $T_* > 0$. Since $R : t \longmapsto v_r(t)B_u z_u(t-r)$ is continuous in $[0,T_*]$, there exists a sequence $(R_n) \subset C^1([0,T_*], H_u)$ such that $R_n \longrightarrow R$ in $C([0,T_*], H_u)$. Furthermore, for any $z_u(0) \in H_u$, one can find a sequence $(x_n) \subset \mathcal{D}(A_u)$ such that $x_n \longrightarrow z_u(0)$ as $n \longrightarrow +\infty$ in H_u (the existence of such a sequence (x_n) is lawful since the operator A_u generates a semigroup of contractions $S_u(t)$ in H_u, which implies $\overline{\mathcal{D}(A_u)} = H_u$). Let $(z_{u,n}) \subset C([0,T_*], H_u)$ be such that

$$z_{u,n}(t) := S_u(t)x_n + \int_0^t S_u(t-\sigma)R_n(\sigma)\,\mathrm{d}\sigma,\ \ t\in[0,\ T_*], \tag{17}$$

the unique classical solution of the system

$$\begin{cases} \dfrac{\mathrm{d}z_{u,n}(t)}{\mathrm{d}t} = A_u z_{u,n}(t) + R_n(t), & t\in[0,\ T_*],\\ z_{u,n}(0) = x_n. \end{cases} \tag{18}$$

That is $z_{u,n}(t) \in \mathcal{D}(A_u)$ and the function $t \longmapsto z_{u,n}(t)$ is continuously differentiable in $[0,T_*]$ (see [16], Theorem 1.5, p. 187). Let us now show that $z_{u,n} \longrightarrow z_u$ as $n \longrightarrow +\infty$ in $(C([0,\ T_*], H_u);\ \|.\|_\infty)$. It follows from (16) and (17) with the fact that $S_u(t)$ is a semigroup of contractions, that for each $t \in [0,T_*]$, we have

$$\|z_{u,n}(t) - z_u(t)\| \leqslant \|x_n - z_u(0)\| \\ + T_* \sup_{s\in[0,T_*]} \|R_n(s) - R(s)\| \to 0, \quad \text{as } n \to +\infty. \tag{19}$$

Then $z_{u,n} \to z_u$ as $n \to +\infty$ in $(C([0,T_*], H_u); \|.\|_\infty)$. It follows by multiplying (18) by $z_{u,n}(t)$ and integrating from s to T_* (where $s \in [0,T_*]$), that

$$\|z_{u,n}(T_*)\|^2 - \|z_{u,n}(s)\|^2 \leqslant 2 \int_s^{T_*} \langle R_n(\sigma), z_{u,n}(\sigma)\rangle \mathrm{d}\sigma, \ \ \forall\, s \in [0,\ T_*]. \tag{20}$$

Furthermore, we get, from (20) via the dominated convergence theorem, that

$$\|z_u(T_*)\|^2 - \|z_u(s)\|^2 \leqslant -2\rho \int_s^{T_*} \frac{|\langle B_u z_u(\sigma - r), z_u(\sigma)\rangle|^2}{\|z_u^\sigma\|^2_{C_u}} \mathrm{d}\sigma \\ \leqslant 0, \ \ \forall\, s \in [0,\ T_*]. \tag{21}$$

In other words, $t \longmapsto \|z_u(t)\|$ is a nonincreasing function on $[0,+\infty)$. In particular, from (21), we have

$$\|z_u(t)\| \leqslant \|z_u(0)\| \leqslant \|\varphi_u\|_{C_u}, \ \ \forall\, t \in [0,\ +\infty). \tag{22}$$

Using now the fact that $t \longmapsto \|z_u(t)\|$ is continuous in $[-r,0]$, one deduces that

$$\|z_u(t)\| \leqslant \|\varphi_u\|_{C_u}, \ \ \forall\, t \in [-r,\ 0]. \tag{23}$$

Combining (22) and (23), one gets

$$\|z_u^t\|_{C_u} \leqslant \|\varphi_u\|_{C_u}, \ \ \forall\, t \in [0,\ +\infty). \tag{24}$$

Now we will establish the continuity of z_u^t with respect to φ_u. To this end, let $t \in [0,+\infty)$ be fixed and let $\varphi_u \in C_u$. For any initial function $\widetilde{\varphi}_u \in C_u$, the corresponding solution $y_u(t)$ of (7) verifies

$$z_u(s) - y_u(s) = S_u(s)\left(\varphi_u(0) - \widetilde{\varphi}_u(0)\right) \\ + \int_0^s S_u(s-\tau)\big(G(z_u^\tau) - G(y_u^\tau)\big)\mathrm{d}\tau, \ \ \forall\, s \in [0,\ t].$$

Using the fact that G is Lipschitz continuous and that $S_u(t)$ is a semigroup of contractions, we obtain

$$\|z_u(s) - y_u(s)\| \leqslant \|\varphi_u - \widetilde{\varphi}_u\|_{C_u} + K(\rho, \|B\|) \int_0^s \|z_u^\tau - y_u^\tau\|_{C_u} \mathrm{d}\tau, \ \ \forall\, s \in [0,\ t].$$

It follows from the Gronwall's inequality, that

$$\|z_u^t - y_u^t\|_{C_u} \leqslant \|\varphi_u - \widetilde{\varphi}_u\|_{C_u} e^{K(\rho,\|B\|)t}.$$

Thus the mapping $\varphi_u \longmapsto z_u^t$ is Lipschitz continuous from C_u to C_u.

To show that (8) admits a unique global mild solution let's consider the map $f\colon \mathbb{R}^+ \times C_s \longrightarrow H_s$ defined by

$$f\colon (t,\phi) \longmapsto v_r(t) B_s \phi(-r).$$

It is easy to see that the function f satisfies:

$$\|f(t,\phi_1) - f(t,\phi_2)\| \leqslant \rho\|B\|\|\phi_1 - \phi_2\|_{C_s}, \ \ \forall t \geqslant 0, \ \forall \phi_1, \phi_2 \in C_s.$$

Thus f is globally Lipschitz continuous, so system (8) admits, in $C([-r,\infty), H_s)$, a unique global mild solution given by

$$z_s(t) = S_s(t) z_s(0) + \int\limits_0^t S_s(t-\sigma) v_r(\sigma) B_s z_s(\sigma - r)\,\mathrm{d}\sigma, \ \ \forall\, t \in [0, \ +\infty). \tag{25}$$

Therefore, system (1) possesses a unique global mild solution. Let us now show the desired estimate (13). It follows by the variation of constants formula (16), that

$$\chi(s) = z_u(s) - S_u(s) z_u(0), \ \ \forall\, s \geqslant 0, \tag{26}$$

where

$$\chi(s) = -\rho \int\limits_0^s S_u(s-\sigma) \frac{\langle B_u z_u(\sigma - r), z_u(\sigma)\rangle}{\|z_u^\sigma\|_{C_u}^2} B_u z_u(\sigma - r)\,\mathrm{d}\sigma.$$

It yields by Cauchy-Schwartz's inequality that for any $T > r$, we have

$$\|\chi(s)\| \leqslant \rho\sqrt{T}\|B\| \left(\int\limits_0^s \frac{|\langle B_u z_u(\sigma - r), z_u(\sigma)\rangle|^2}{\|z_u^\sigma\|_{C_u}^2}\,\mathrm{d}\sigma \right)^{\frac{1}{2}}, \ \ \forall\, s \in [0, \ T]. \tag{27}$$

In view of

$$\begin{aligned} \langle B_u S_u(\sigma - r) z_u(0), S_u(\sigma) z_u(0)\rangle &= \langle B_u S_u(\sigma - r) z_u(0) - B_u z_u(\sigma - r), z_u(\sigma)\rangle \\ &+ \langle B_u z_u(\sigma - r), z_u(\sigma)\rangle - \langle B_u S_u(\sigma - r) z_u(0), \chi(\sigma)\rangle, \ \forall\, \sigma \in [r, T], \end{aligned}$$

we obtain

$$|\langle B_u S_u(\sigma - r) z_u(0), S_u(\sigma) z_u(0)\rangle| \leqslant \|B\| \|z_u(0)\| \|\chi(\sigma - r)\| \\ + \|B\| \|z_u(0)\| \|\chi(\sigma)\| + |\langle Bz(\sigma - r), z(\sigma)\rangle|.$$

Using (27) one easily gets

$$\begin{aligned} &|\langle B_u S_u(\sigma - r) z_u(0), S_u(\sigma) z_u(0)\rangle| \\ &\quad \leqslant 2\rho\sqrt{T}\|B\|^2 \|z_u(0)\| \left(\int_0^T \frac{|\langle B_u z_u(\sigma - r), z_u(\sigma)\rangle|^2}{\|z_u^\sigma\|_{C_u}^2} \mathrm{d}\sigma \right)^{\frac{1}{2}} \\ &\quad + |\langle B_u z_u(\sigma - r), z_u(\sigma)\rangle|. \end{aligned} \tag{28}$$

Replacing $z_u(0)$ by $z_u(t)$, $\forall\, t \geqslant 0$ in (28), we obtain

$$\begin{aligned} &|\langle B_u S_u(\sigma - r) z_u(t), S_u(\sigma) z_u(t)\rangle| \\ &\quad \leqslant 2\rho\sqrt{T}\|B\|^2 \|z_u(t)\| \left(\int_t^{t+T} \frac{|\langle B_u z_u(\sigma - r), z_u(\sigma)\rangle|^2}{\|z_u^\sigma\|_{C_u}^2} \mathrm{d}\sigma \right)^{\frac{1}{2}} \\ &\quad + |\langle B_u z_u(\sigma + t - r), z_u(\sigma + t)\rangle|. \end{aligned}$$

Integrating the last inequality over the interval $[r,\ T]$, and using the Schwartz's inequality with the fact that

$$\|z_u^\sigma\|_{C_u} \leqslant \|z_u(t)\|, \quad \forall\, \sigma \in [t+r,\ t+T],$$

lead to

$$\begin{aligned} &\int_r^T |\langle B_u S_u(\sigma - r) z_u(t), S_u(\sigma) z_u(t)\rangle| \mathrm{d}\sigma \\ &\quad \leqslant 2\rho T^{\frac{1}{2}} (T - r) \|B\|^2 \|z_u(t)\| \left(\int_t^{t+T} \frac{|\langle B_u z_u(\sigma - r), z_u(\sigma)\rangle|^2}{\|z_u^\sigma\|_{C_u}^2} \mathrm{d}\sigma \right)^{\frac{1}{2}} \\ &\quad + (T - r)^{\frac{1}{2}} \|z_u(t)\| \left(\int_{t+r}^{t+T} \frac{|\langle B_u z_u(\sigma - r), z_u(\sigma)\rangle|^2}{\|z_u^\sigma\|_{C_u}^2} \mathrm{d}\sigma \right)^{\frac{1}{2}} \\ &\quad \leqslant C_{**}(r) \|z_u(t)\| \left(\int_t^{t+T} \frac{|\langle B_u z_u(\sigma - r), z_u(\sigma)\rangle|^2}{\|z_u^\sigma\|_{C_u}^2} \mathrm{d}\sigma \right)^{\frac{1}{2}}, \quad \forall\, t \geqslant 0, \end{aligned} \tag{29}$$

where $C_{**}(r) := (1 + 2\rho T^{\frac{1}{2}}(T-r)^{\frac{1}{2}}\|B\|^2)(T-r)^{\frac{1}{2}} > 0$. This ends the proof of Theorem 1. □

Remark 2

1. *In the proof of Theorem 1,* $\dim H_u$ *can be finite or infinite.*
2. *System (1) admits a unique global mild solution for any* $\rho > 0$.
3. *If* $\dim H_u < +\infty$*, one can easily see that* $z_u \in C^1([-r, +\infty), H_u)$.

4. Exponential stabilization

In the sequel, we suppose that the state space H is decomposable according to (2) with $\dim H_u < +\infty$.

To establish our main result the following lemma will be needed.

Lemma 1 *Let* H *be a finite dimensional Hilbert space. Then, the following assertions*

$$\langle BS(t-r)\phi, S(t)\phi\rangle = 0, \quad \forall\, t \geqslant r \Rightarrow \phi = 0, \tag{30}$$

and

$$\int_r^T \left|\langle BS(t-r)\phi, S(t)\phi\rangle\right| \mathrm{d}t \geqslant \delta_T(r)\|\phi\|^2, \quad \forall\, \phi \in H,$$

$$(\textit{for some } \ T > r \ \textit{ and } \ \delta_T(r) > 0), \tag{31}$$

are equivalent.

Proof. The proof is based on the fact that for ϕ given in H the mapping $t \mapsto \langle BS(t-r)\phi, S(t)\phi\rangle$ is real analytic on $[r, +\infty[$ and so it vanishes for all $t \geqslant r$ as soon as it vanishes on some interval (r, T) (see e.g. [10], Corollary 1.2.6, p. 14). For the rest of proof see [6]. □

The next result provides a sufficient condition for the uniform exponential stabilizability of system (1) with an explicit decay estimate of the stabilized state using the feedback control (12).

Theorem 2 *Let* A *generate a semigroup of contractions* $S(t)$ *on* H *such that condition (5) holds, and assume* A *allows the decomposition (2) of* H *with* $\dim H_u < +\infty$*. Let* $B \in \mathcal{L}(H)$ *satisfy (4) such that condition (6) holds. Then, there exists* $\rho > 0$ *such that the feedback control (12) uniformly exponentially stabilizes system (1).*

Proof. Since $\dim H_u < +\infty$, from Lemma 1 and the assumption (6), we have (for some $T > r$ and $\delta_T(r) > 0$),

$$\int_r^T |\langle B_u S_u(\sigma - r)z_u(\sigma), S_u(\sigma)z_u(\sigma)\rangle| \mathrm{d}\sigma \geqslant \delta_T(r)\|z_u(t)\|^2, \quad \forall\, \phi \in H_u. \quad (32)$$

It follows from (21) that the solution of system (7) satisfies

$$\begin{aligned} \|z_u(t+T)\|^2 - \|z_u(t)\|^2 &\leqslant -2\rho \int_t^{t+T} \frac{|\langle B_u z_u(\sigma - r), z_u(\sigma)\rangle|^2}{\|z_u^\sigma\|^2_{C_u}} \mathrm{d}\sigma \\ &\leqslant 0, \quad \forall\, t \geqslant 0. \end{aligned} \quad (33)$$

It yields from (29), that

$$\begin{aligned} &\|z_u(t+T)\|^2 - \|z_u(t)\|^2 \leqslant \\ &\qquad \leqslant \frac{-2\rho}{C^2_{**}(r)} \|z_u(t)\|^{-2} \left(\int_r^T |\langle B_u S_u(\sigma - r)z_u(t), S_u(\sigma)z_u(t)\rangle| \mathrm{d}\sigma \right)^2. \end{aligned} \quad (34)$$

Combining (32) with the fact that $t \longmapsto \|z_u(t)\|$ is a decreasing function, we obtain from estimate (34) that

$$\|z_u(t+T)\|^2 - \|z_u(t)\|^2 \leqslant -\frac{2\rho\delta_T^2(r)}{C^2_{**}(r)} \|z_u(t+T)\|^2,$$

which implies that

$$\left(1 + \frac{2\rho\delta^2}{C^2_{**}(r)}\right) \|z_u(t+T)\|^2 \leqslant \|z_u(t)\|^2, \ \forall t \geqslant 0.$$

Thus

$$\|z_u((k+1)T)\|^2 \leqslant \frac{\|z_u(kT)\|^2}{1 + \dfrac{2\rho\delta_T^2(r)}{C^2_{**}(r)}}, \qquad k \in \mathbb{N}.$$

Hence, $\|z_u(kT)\|^2 \leqslant \dfrac{1}{\beta^k(r,\rho)} \|z(0)\|^2$, where $\beta(r,\rho) := 1 + \dfrac{2\rho\delta_T^2(r)}{C^2_{**}(r)} > 1$. Let us now set $k = \left[\dfrac{t}{T}\right]$ $\left(\text{where } \left[\dfrac{t}{T}\right] \text{ designates the integer part of } \dfrac{t}{T}\right)$. Since the mapping $t \longmapsto \|z_u(t)\|$ decreases for $t \geqslant 0$, we obtain the uniform exponential decay

$$\|z_u(t)\| \leqslant \sqrt{\beta(r,\rho)}\|\varphi_u\|_{C_u} e^{\frac{-\ln(\beta(r,\rho))}{2T}t}, \quad \forall\, t \geqslant 0. \tag{35}$$

Since the map $t \longmapsto \|z_s(t)\|$ is continuous on $[-r,\ 0]$, then

$$\|z_s(t)\| \leqslant \|\varphi_s\|_{C_s}, \quad \forall\, t \in [-r,\ 0]. \tag{36}$$

Furthermore, according to (5) that the semigroup $S_s(t)$ satisfies the inequality

$$\|S_s(t)\| \leqslant M e^{-\gamma_* t}, \quad \forall\, t \geqslant 0 \quad (\text{for some } \ M, \gamma_* > 0), \tag{37}$$

(see [20]). It follows from (25) by using (36) and (37), that

$$\|z_s(t)\| \leqslant M\|z_s(0)\|e^{-\gamma_* t} + \rho M\|B\|^2 e^{\gamma_* r} e^{-\gamma_* t} \int\limits_{-r}^{t-r} e^{\gamma_* \tau}\|z_s(\tau)\|\mathrm{d}\tau,$$

which implies that

$$\begin{aligned} e^{\gamma_* t}\|z_s(t)\| &\leqslant M\|z_s(0)\| + \rho M\|B\|^2 e^{\gamma_* r} \int\limits_{-r}^{t} e^{\gamma_* \tau}\|z_s(\tau)\|\mathrm{d}\tau \\ &\leqslant M(1+\rho r\|B\|^2 e^{\gamma_* r})\|\varphi_s\|_{C_s} + \rho M\|B\|^2 e^{\gamma_* r} \int\limits_{0}^{t} e^{\gamma_* \tau}\|z_s(\tau)\|\mathrm{d}\tau. \end{aligned} \tag{38}$$

Taking $\kappa(t) = e^{\gamma_* t}\|z_s(t)\|$. Then, the Gronwall's inequality gives

$$\kappa(t) \leqslant M(1+\rho r\|B\|^2 e^{\gamma_* r})\|\varphi_s\|_{C_s} e^{\rho M\|B\|^2 e^{\gamma_* r} t}, \quad \forall\, t \geqslant 0.$$

Hence,

$$\|z_s(t)\| \leqslant M(1+\rho r\|B\|^2 e^{\gamma_* r})\|\varphi_s\|_{C_s} e^{-K(\rho,\gamma_*)t}, \quad \forall\, t \geqslant 0, \tag{39}$$

where $K(\rho,\gamma_*) := \gamma_* - \rho M\|B\|^2 e^{\gamma_* r}$. Taking $\rho \in \left(0, \dfrac{\gamma_*}{M\|B\|^2 e^{\gamma_* r}}\right)$, we get $K(\rho,\gamma_*) > 0$. Therefore, one deduce from (36) and (39) that

$$\|z_s^t\|_{C_s} \leqslant \max\left\{1, M(1+\rho r\|B\|^2 e^{\gamma_* r})\right\}\|\varphi_s\|_{C_s}, \ \forall\, t \geqslant 0. \tag{40}$$

From inequalities (39) and (35), we deduce that the solution of system (1) satisfies

$$\|z(t)\| \leqslant \mathcal{H}\|\varphi\|_C e^{-\sigma t}, \quad \forall\, t \geqslant 0, \tag{41}$$

where $\mathcal{H} = \mathcal{H}(\rho,\gamma_*,r,\|B\|,M,T,\delta_T) := \max\{M(1+\rho r\|B\|^2 e^{\gamma_* r}), \sqrt{\beta(r,\rho)}\}$ and $\sigma := \min\left\{K(\rho,\gamma_*), \dfrac{\ln(\beta(r,\rho))}{2T}\right\}$. This ends the proof of Theorem 2. □

Remark 3

1. *The feedback control* $v_r(t)$ *is bounded and we have for each* $t \geqslant 0$, $|v_r(t)| \leqslant \rho\|B\|$.

2. *The exponential convergence rate* σ *depends on* r, ρ, T, M, γ_* *and* $\delta_T(r) = \inf\limits_{\|z\|=1} \int\limits_r^T |\langle B_u S_u(t-r)z, S_u(t)z\rangle| \mathrm{d}t$.

3. *Note that* $\dim H_u = 0$, *implies* $z_u(t) = 0$, $\forall\, t \geqslant 0$. *Thus* $z(t) = z_s(t) = S_s(t)z_s(0)$. *Therefore* $\|z(t)\| \leqslant M\|z_s(0)\|e^{-\gamma_* t}$, $\forall t \geqslant 0$. *Hence system* (1) *is uniformly exponentially stable.*

4. *The fact that the function* $t \longmapsto \|z_u(t)\|$ *is nonincreasing on* $[0, +\infty)$, *implies* $\|z_u^t\|_{C_u} = \|z_u(t-r)\|$, $\forall t \geqslant r$. *Therefore, the feedback control (12) can be expressed as:*

$$v_r(t) = \begin{cases} -\rho \dfrac{\langle B_u z_u(t-r), z_u(t)\rangle}{\|z_u(t-r)\|^2}, & t \in E_1, \\ -\rho \dfrac{\langle B_u z_u(t-r), z_u(t)\rangle}{\|\varphi_u\|^2_{C([t-r,0],H_u)}}, & t \in E_2\,, \\ 0, & t \in \mathbb{R}_+ \backslash (E_1 \cup E_2), \end{cases} \tag{42}$$

where $E_1 = \{t \geqslant r;\ z_u(t-r) \neq 0\}$, $E_2 = \{t \in [0,r];\ \|\varphi_u\|_{C([t-r,0],H_u)} \neq 0\}$ *and* $\rho > 0$.

5. *The feedback control* (12) *depends only on the unstable part* $z_u(t)$.

6. *Since the function* $t \longmapsto \|z_u(t)\|$, $\forall t \geqslant 0$, *decreases, if there is a* $t^* \geqslant 0$ *such that* $z_u(t^*) = 0$ *then* $z_u(t) = 0$, $\forall t \geqslant t^*$. *In other word, if there is a* $t^* \geqslant r$ *such that* $z_u(t^*) = 0$, *then for each* $t \geqslant t^*$, *we get* $v_r(t) = 0$ *and so, in this case, we have* $\|z(t)\| = \|z_s(t)\| \leqslant Me^{-\gamma_*(t-t^*)}$, $\forall\, t \geqslant t^*$, *i.e. system (1) is uniformly exponentially stable.*

7. *If there is a* $t^* \geqslant r$ *such that* $z_s(t^*) = 0$ *then* $z_s(t) = 0$, $\forall\, t \geqslant t^* \geqslant r$. *Indeed, in this case the solution of (8) is written as*

$$z_s(t) = \int\limits_{t^*}^{t} S_s(t-\tau)v_r(\tau)B_s z_s(\tau - r)\mathrm{d}\tau, \quad \forall\, t \geqslant t^* \geqslant r. \tag{43}$$

Thus

$$e^{\gamma_* t}\|z_s(t)\| \leqslant \rho M\|B\|^2 e^{\gamma_* r}\int_0^t e^{\gamma_*\tau}\|z_s(\tau)\|\,\mathrm{d}\tau,\quad (M,\rho>0).$$

Using Gronwall's inequality gives $z_s(t)=0$, $\forall t\geqslant t^*\geqslant r$.

8. *Combining points 6 and 7, we deduce that if there is a* $t^*\geqslant r$ *such that* $z(t^*)=0$ *then* $z(t)=0$, $\forall t\geqslant t^*\geqslant r$.

9. *A typical situation where the above decomposition (2) holds is the case where the operator A is self-adjoint with compact resolvent. In this case we have only finitely many eigenvalues* λ *such that* $\mathrm{Re}(\lambda)\geqslant -\gamma$, $(\gamma>0)$ *each with finite-dimensional eigenspace (see [19]), and ordered so that* $\mathrm{Re}(\lambda_1)\geqslant \mathrm{Re}(\lambda_2)\geqslant\ldots$ *Let's denote by* λ_N *the first eigenvalue with negative real part. The eigenvectors* $(\phi_j)_{1\leqslant j\leqslant r_n}$ *associated to* λ_n *(*r_n *denotes the multiplicity of* λ_n*) form a complete set in H, so for all* $z\in H$, *we have*

$$z=\sum_{n=1}^{N}\sum_{j=1}^{r_n}\langle z,\phi_{n_j}\rangle\phi_{n_j}+\sum_{n=N+1}^{+\infty}\sum_{j=1}^{r_n}\langle z,\phi_{n_j}\rangle\phi_{n_j}.$$

The state space H can be decomposed according to $H=H_u+H_s$, *where* $H_u=\mathit{span}\{\phi_{n_j},\ 1\leqslant n\leqslant N,\ 1\leqslant j\leqslant r_n\}$ *so that* $\dim H_u=\sum_{j=1}^{N} r_j$. *For* $z\in\mathcal{D}(A)=\left\{z\in H;\ \sum_{n=1}^{+\infty}|\lambda_n|^2\sum_{j=1}^{r_n}|\langle z,\phi_{n_j}\rangle|^2<+\infty\right\}$, *we have*

$$Az=\sum_{n=1}^{+\infty}\lambda_n\sum_{j=1}^{r_n}\langle z,\phi_{n_j}\rangle\phi_{n_j}=A_u z+A_s z,$$

where $A_u z=\sum_{n=1}^{N}\lambda_n\sum_{j=1}^{r_n}\langle z,\phi_{n_j}\rangle\phi_{n_j}$ *and* $A_s z=\sum_{n=N+1}^{+\infty}\lambda_n\sum_{j=1}^{r_n}\langle z,\phi_{n_j}\rangle\phi_{n_j}$. *In this case, the decomposition assumption and the growth assumption (5) hold.*

10. *Once the decomposition (2) is carried out, the control operator B admits a decomposition of type (4) provided both subspaces* H_u *and* H_s *are invariant under B. This holds for example for the operator B defined by* $B=\alpha I+\beta P_u$ *where I is the identity operator and* α *and* β *some real constants. A sufficient condition for B to possess such a decomposition is that B commutes with A, that is* $B\mathcal{D}(A)\subset\mathcal{D}(A)$ *and* $AB=BA$.

5. Applications

In this section, we give two illustrating examples of the previous established results.

Example 1 Let us consider the following fourth-order partial functional differential equation:

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} = -\dfrac{\partial^4 z(x,t)}{\partial x^4} - \dfrac{\partial^2 z(x,t)}{\partial x^2} + v(t)Bz\left(x, t-\dfrac{3}{2}\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in (-\pi,\pi)\times(0,+\infty), \\ z(x,t) = 2\sin(x) - \sin(2x), \qquad (x,t)\in(-\pi,\pi)\times\left[-\dfrac{3}{2},\ 0\right]. \end{cases} \tag{44}$$

Here $\varphi_u(x,t) = 2\sin(x)$ and $\varphi_s(x,t) = -\sin(2x)$. The sate space H is defined by $H = L^2(-\pi,\pi)$, the operator A is given by $Az = -\dfrac{\partial^4 z(x,t)}{\partial x^4} - \dfrac{\partial^2 z(x,t)}{\partial x^2}$, for $z \in \mathcal{D}(A) = \{z \in H^4(-\pi,\pi);\ \dfrac{\partial^n z}{\partial x^n}(-\pi) = \dfrac{\partial^n z}{\partial x^n}(\pi),\ \ n = 0,\ldots,3\}$, is an infinitesimal generator of a semigroup of contractions $S(t)$ defined by $S(t)z = \sum_{j=1}^{+\infty} e^{\lambda_j t}\langle z,\phi_j\rangle\phi_j$, with eigenvalues explicitly given by $\lambda_j = -j^4 + j^2$, $(j\in\mathbb{N}^*)$, associated to the eigenfunctions $\phi_j(x) = \dfrac{1}{\sqrt{\pi}}\sin(jx), \forall j \geqslant 1$ (see [14]). In this case the subspace $H_u = \operatorname{span}\{\phi_1\}$ and $S_u(t) = I_{H_u}$ (identity operator). Let's take, for instance, $B = I_H$, so $B_u = I_{H_u}$ and $B_s = I_{H_s}$. The solution of (44) can be written as

$$z(x,t) = \sum_{j=1}^{+\infty} a_j(t)\phi_j(x) = \sum_{j=1}^{+\infty}\langle z(t),\phi_j\rangle_{L^2(-\pi,\pi)}\phi_j(x),\ \ \forall t \geqslant 0,$$

and $z_u(.,t) = a_1(t)\phi_1 \in H_u$, $\forall t \geqslant 0$. Furthermore, we have

$$\left\langle B_u S_u\left(t-\frac{3}{2}\right)\phi, S_u(t)\phi\right\rangle = \|\phi\|^2,\ \ \forall\,\phi\in H_u,\ \ \forall t \geqslant \frac{3}{2}.$$

Then, hypothesis (6) is verified. Note that in the case where $\|z_u^{t^*}\|_{C_u} = 0$, for some $t^* \geqslant 0$, we can argue, using Remark 3, that system (44) is uniformly exponentially stable. Otherwise, we will show that $a_1(t) \geqslant 0$, $\forall t \geqslant 0$. To this end, multiplying system (7) associated to (44), i.e.,

$$a_1'(t) = -\rho \frac{a_1(t)a_1^2\left(t-\dfrac{3}{2}\right)}{\sup\limits_{s\in[t-\frac{3}{2},t]} a_1^2(s)}, \quad \forall t \geqslant 0, \tag{45}$$

by $a_1^- := \min\{a_1, 0\}$, gives

$$\frac{\mathrm{d}(a_1^-(t))^2}{\mathrm{d}t} = -2\rho \frac{(a_1^-(t))^2 a_1^2\left(t-\dfrac{3}{2}\right)}{\sup\limits_{s\in[t-\frac{3}{2},t]} a_1^2(s)} \leqslant 0, \quad \forall\, t \geqslant 0,$$

where $\rho > 0$. Then, we get $(a_1^-(t))^2 \leqslant (a_1^-(0))^2$, $\forall t \geqslant 0$. Since $a_1(0) = 2\sqrt{\pi}$, we deduce that $a_1^-(t) = 0$, $\forall t \geqslant 0$, and hence $a_1(t) \geqslant 0$, $\forall t \geqslant 0$. Then the feedback control (12) is given by:

$$v_{\frac{3}{2}}(t) = \begin{cases} -\rho \dfrac{a_1(t)}{a_1\left(t-\dfrac{3}{2}\right)}, & t \geqslant \dfrac{3}{2}, \\ -\dfrac{\rho}{2\sqrt{\pi}} a_1(t), & t \in \left[0, \dfrac{3}{2}\right]. \end{cases} \tag{46}$$

Moreover, the function a_1 satisfies:

$$a_1'(t) = \begin{cases} -\rho a_1(t), & t \geqslant \dfrac{3}{2}, \\ -\rho a_1(t), & t \in \left[0, \dfrac{3}{2}\right]. \end{cases} \tag{47}$$

In view of (47), we infer that

$$a_1(t) = 2\sqrt{\pi} e^{-\rho t}, \quad t \geqslant 0. \tag{48}$$

Therfore

$$\|z_u(t)\| \leqslant 2\sqrt{\pi} e^{-\rho t}, \quad t \geqslant 0. \tag{49}$$

The second component $z_s(t)$ satisfies

$$z_s(t) = \sum_{j=2}^{+\infty} e^{\lambda_j t} \langle z_s(0), \phi_j \rangle \phi_j + \left(\int_0^t v_{\frac{3}{2}}(\sigma) \sum_{j=2}^{+\infty} e^{\lambda_j (t-\sigma)} \left\langle z_s\left(\sigma - \frac{3}{2}\right), \phi_j \right\rangle \phi_j \,\mathrm{d}\sigma \right).$$

Moreover, using the fact that $\lambda_j = -j^4 + j^2 < -1$, for any $j \geqslant 2$, we deduce that

$$
\begin{aligned}
\|z_s(t)\| &\leqslant e^{-t}\|z_s(0)\| + \rho e^{\frac{3}{2}} e^{-t} \int_{-\frac{3}{2}}^{t} e^{\sigma}\|z_s(\sigma)\|\mathrm{d}\sigma \\
&\leqslant e^{-t}\left(1 + \frac{3\rho}{2} e^{\frac{3}{2}}\right)\|\varphi_s\|_{C_s} + \rho e^{\frac{3}{2}} e^{-t} \int_{0}^{t} e^{\sigma}\|z_s(\sigma)\|\mathrm{d}\sigma, \quad \forall\, t \geqslant 0.
\end{aligned}
$$

The last inequality implies that

$$
e^{t}\|z_s(t)\| \leqslant \left(1 + \frac{3\rho}{2} e^{\frac{3}{2}}\right) + \rho e^{\frac{3}{2}} \int_{0}^{t} e^{\sigma}\|z_s(\sigma)\|\mathrm{d}\sigma.
$$

From Gronwall's inequality, one gets

$$
\|z_s(t)\| \leqslant \frac{5}{2} e^{-\rho t}, \quad \forall t \geqslant 0, \tag{50}
$$

where $\rho \in \left(0, \dfrac{1}{1 + e^{\frac{3}{2}}}\right)$. Combining (49) and (50) we obtain

$$
\|z(t)\|_{L^2(-\pi,\pi)} \leqslant (2\sqrt{\pi} + \frac{5}{2}) e^{-\rho t}, \quad \forall t \geqslant 0, \tag{51}
$$

where ρ is small enough.

Example 2 Let us consider the system defined by

$$
\begin{cases}
\dfrac{\partial z(x,t)}{\partial t} = \dfrac{\partial^2 z(x,t)}{\partial x^2} + \pi^2 \left(\displaystyle\int_0^1 \phi_1(x) z(x,t)\mathrm{d}x \right) \phi_1 + v(t) z(x, t-1) \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for } (x,t) \in (0,1) \times (0, +\infty), \\
z(x,t) = 3 + \cos(\pi x) + \cos(2\pi x) \qquad \text{for } (x,t) \in (0,1) \times [-1, 0],
\end{cases} \tag{52}
$$

where $\phi_1(x) = \sqrt{2}\cos(\pi x)$. The state space is $H = L^2(0,1)$ and $Az = \dfrac{\partial^2 z}{\partial x^2} + \pi^2 \left(\displaystyle\int_0^1 \phi_1(x) z(x,t)\mathrm{d}x \right) \phi_1$, for all $z \in \mathcal{D}(A) = \{z \in H^2(0,1);\ z'(0) = z'(1) = 0\}$.

The spectrum of A is given by the eigenvalues $\lambda_0 = \lambda_1 = 0$, $\lambda_j = -\pi^2 j^2$, $j \geqslant 2$, and eigenfunctions $\phi_0(x) = 1$ and $\phi_j(x) = \sqrt{2}\cos(j\pi x)$ for all $j \geqslant 1$. The operator A generates the semigroup of contractions $S(t)y = \sum_{j=0}^{\infty} e^{\lambda_j t}\langle y, \phi_j\rangle\phi_j$. Furthermore, the solution of (52) can be written as:

$$z(x,t) = \sum_{j=0}^{+\infty} a_j(t)\phi_j(x) = \sum_{j=0}^{+\infty}\langle z(t), \phi_j\rangle_{L^2(0,1)}\phi_j(x), \quad \forall t \geqslant 0,$$

and $z_u(.,t)=a_0(t)\phi_0+a_1(t)\phi_1\in H_u$, $\forall t\geqslant 0$, where $a_0, a_1 \in C([0,+\infty), \mathbb{R})$. Then we have

$$\langle B_u S_u(t-1)y, S_u(t)y\rangle = |\langle y, \phi_0\rangle|^2 + |\langle y, \phi_1\rangle|^2, \quad \forall y \in H_u = \operatorname{span}\{\phi_0, \phi_1\}.$$

It is easy to see that the assumption (6) is satisfied and the hypotheses of Theorem 2 are all fulfilled. Using the same argument as in the Example 1, if there exists $t^* \geqslant 0$ such that $\|z_u^{t^*}\|_{C_u} = 0$, then system (52) is uniformly exponentially stable. Otherwise, the feedback control (12) given by:

$$v_1(t) = \begin{cases} -\rho\dfrac{a_0^2(t) + a_1^2(t)}{a_0^2(t-1) + a_1^2(t-1)}, & t \geqslant 1, \\ -\dfrac{2\rho}{19}\left(a_0^2(t) + a_1^2(t)\right), & t \in [0,\ 1], \end{cases} \tag{53}$$

uniformly exponentially stabilizes system (52).

Example 3 The following system models the heat-transfer in the square domain $\Omega = (0,1) \times (0,1)$

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,z) = \Delta z(x,z) + v(t)z(x,t-r) & (x,t) \in \Omega \times (0,+\infty), \\ \dfrac{\partial z}{\partial \nu}(x,z) = 0 & (x,t) \in \partial\Omega \times (0,+\infty), \\ z(x,t) = \varphi(t) & (x,t) \in \Omega \times [-r,0]. \end{cases} \tag{54}$$

Here the term $v(t)z(x,t-r)$ describes the heat exchange between the domain Ω and a surrounding medium of zero temperature in accordance with Newton's Law (e.g., [18]). The heat transfer coefficient v is regarded as a bilinear control. System (54) can be cast into the form (1) by setting $z(t) := z(.,t) \in H$ where $H = L^2(\Omega)$ and $B := I$ (Identity operator). Moreover, the operator A is taken as the Laplace operator with domain $\mathcal{D}(A) = \{y \in H^2(\Omega) : \dfrac{\partial y}{\partial \nu} = 0 \text{ on } \partial\Omega\}$. Operator A admits a basis of H made of the following eigenfunctions

$$\Phi_{n,m}(x_1, x_2) = \cos n\pi x_1 \, \cos m\pi x_2, \quad n, m \in \mathbb{N},$$

respectively associated to the eigenvalues $\lambda_{n,m} = -(n^2 + m^2)\pi^2$. Note that for $n = m = 0$ we have $\Phi_{0,0} = 1$ and $\lambda_{0,0} = 0$. Hence, the space H admits the decomposition

$$H = H_u \oplus H_s \,,$$

where $H_u = \operatorname{span}\{\Phi_{0,0}\}$, $H_s = \operatorname{span}\left\{\Phi_{n,m} : (n, m) \neq (0, 0)\right\}$ and the semi-group $S_u(t)$ is the identity operator. Moreover we have

$$\langle B_u S_u(t - r)\phi, S_u(t)\phi\rangle = \|\phi\|^2, \quad \forall\, t \geqslant r \text{ and } \phi \in H_u.$$

Thus, condition (6) is fullfilled. Therefore, the following feedback control exponentially steers the temperature $z(x, t)$ to zero as t tends to infinity:

$$u_r(t) = \begin{cases} -\rho \dfrac{\displaystyle\int_\Omega z_u(x, t - r) z_u(x, t)\,\mathrm{d}x}{\|z_u^t\|^2_{C_u}} & \text{for } t \geqslant 0 \text{ s.t. } \|z_u^t\|_{C_u} \neq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

After simple calculations, the expression of the feedback control can be expressed in terms of the state function z as follows

$$u_r(t) = \begin{cases} -\rho \dfrac{\displaystyle\int_\Omega z(x, t - r)\,\mathrm{d}x \int_\Omega z(x, t)\,\mathrm{d}x}{\|z_u^t\|^2_{C_u}} & \text{for } t \geqslant 0 \text{ s.t. } \|z_u^t\|_{C_u} \neq 0, \\ 0 & \text{elsewhere,} \end{cases}$$

and $\|z_u^t\|^2_{C_u}$ is given by

$$\|z_u^t\|^2_{C_u} = \begin{cases} \left(\displaystyle\int_\Omega z(x, t - r)\,\mathrm{d}x\right)^2 & \text{for } t \geqslant r, \\ \displaystyle\sup_{t - r \leqslant s \leqslant 0} \left(\int_\Omega \varphi(x, s)\,\mathrm{d}x\right)^2 & \text{for } t < r. \end{cases}$$

6. Conclusion

In this work, we have proposed a bounded feedback control which depends only on the state projection on an appropriate finite dimensional subspace to study the uniform exponential stabilization for a class of distributed bilinear parabolic systems with time delay under a weaker non-standard observability condition. The rate of exponential convergence is explicitly given. Three examples are also provided to illustrate the theoretical results.

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