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Dedicated to Professor M.P. Kaźmierkowski
on the occasion of his 70th birthday

Minimum energy control of fractional positive continuous-time linear systems

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Abstract. The minimum energy control problem for the fractional positive continuous-time linear systems is formulated and solved. Sufficient conditions for the existence of solution to the problem are established. A procedure for solving of the problem is proposed and illustrated by a numerical example.

Key words: fractional system, positive system, minimum energy control, procedure.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [1, 2]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Mathematical fundamentals of the fractional calculus are given in the monographs [3–5]. The positive fractional linear systems have been investigated in [6–9]. Stability of fractional linear 1D discrete-time and continuous-time systems has been investigated in the papers [9, 10, 11] and of 2D fractional positive linear systems in [12]. The notion of practical stability of positive fractional discrete-time linear systems has been introduced in [13]. Some recent interesting results in fractional systems theory and its applications can be found in [14–17]. The minimum energy control problem for standard linear systems has been formulated and solved by J. Klamka [18–20] and for 2D linear systems with variable coefficient in [21]. The controllability and minimum energy control problem of fractional discrete-time linear systems has been investigated by Klamka in [22].

In this paper the minimum energy control problem for the fractional positive continuous-time linear systems is formulated and solved. Sufficient conditions for the existence of solution to the problem are established and a procedure for solving of the problem is proposed.

The paper is organized as follows. In Sec. 2 some definitions and theorems concerning fractional positive continuous-time linear systems are recalled. Necessary and sufficient conditions for the reachability of the fractional positive systems are established and the minimum energy control problem is formulated in Sec. 3. The solution of the problem is given in Sec. 4. A procedure for solving the minimum energy control problem and illustrating numerical example are given in Sec. 5. Concluding remarks are given in Sec. 6.

The following notation is used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n \times n$ identity matrix.

2. Preliminaries

The following Caputo definition of the fractional derivative is used [4, 5, 9]

$$D^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}}{(t-\tau)^{\alpha+1-n}} d\tau, \quad n-1<\alpha\le n \in N=\{1,2,...\}, \tag{1}$$

where $\alpha \in \Re$ is the order of fractional derivative and $f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n}$ and $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is the gamma function.

Consider the continuous-time fractional linear system described by the state equation

$$D^\alpha x(t) = Ax(t) + Bu(t), \quad 0<\alpha\le 1, \tag{2}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and input vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$.

Theorem 1. [9] The solution of Eq. (2) is given by

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \quad x(0)=x_0, \tag{3}$$

where
$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^\infty \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{4}$$

$$\Phi(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{5}$$

and $E_\alpha(At^\alpha)$ is the Mittage-Leffler matrix function [13].

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Definition 1. [9] The fractional system (2) is called the (internally) positive fractional system if and only if $x(t) \in \Re_+^n$ and $y(t) \in \Re_+^p$ for $t \geq 0$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \geq 0$.

Theorem 2. [9] The continuous-time fractional system (2) is (internally) positive if and only if the matrix A is a Metzler matrix and

$$A \in M_n, \quad B \in \Re_+^{n \times m}. \tag{6}$$

Lemma 1. The Mittag-Leffler matrix function (4) satisfies the equation

$$\frac{d^\alpha \Phi_0(t)}{dt^\alpha} = A\Phi_0(t). \tag{7}$$

Proof. From (2) and (3) for $Bu(t) = 0$ we have

$$\frac{d^\alpha \Phi_0(t)x_0}{dt^\alpha} = A\Phi_0(t)x_0. \tag{8}$$

Therefore, the equality (7) holds since the equation (8) is satisfied for arbitrary $x_0 \neq 0$.

3. Reachability and problem formulation

Definition 2. The state $x_f \in \Re_+^n$ of the fractional system (2) is called reachable in time t_f if there exist an input $u(t) \in \Re_+^m$, $t \in [0, t_f]$ which steers the state of system (2) from zero initial state $x_0 = 0$ to the state x_f. If every state $x_f \in \Re_+^n$ is reachable in time t_f the system is called reachable in time t_f. If for every state $x_f \in \Re_+^n$ there exist a time t_f such that the state is reachable in time t_f then the system (2) or positive pair (A, B) is called reachable.

A real square matrix is called monomial if each its row and each its column contains only one positive entry and the remaining entries are zero.

Theorem 3. The positive fractional system (2) is reachable in time $t \in [0, t_f]$ if and only if the matrix $A \in M_n$ is diagonal and the matrix $B \in \Re_+^{n \times m}$ is monomial.

Proof. Sufficiency. It is well-known [2, 9] that if $A \in M_n$ is diagonal then $\Phi(t) \in \Re_+^{n \times n}$ is also diagonal and if $B \in \Re_+^{n \times m}$ is monomial then $BB^T \in \Re_+^{n \times n}$ is also monomial. In this case the matrix

$$R_f = \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau)d\tau \in \Re_+^{n \times n} \tag{9}$$

is also monomial and $R_f^{-1} \in \Re_+^{n \times n}$. The input

$$u(t) = B^T\Phi^T(t_f - t)R_f^{-1}x_f \tag{10}$$

steers the state of the system (2) from $x_0 = 0$ to x_f since using (3) for $x_0 = 0$ and (5) we obtain

$$\begin{aligned} x(t_f) &= \int_0^{t_f} \Phi(t_f - \tau)Bu(\tau)d\tau \\ &= \int_0^{t_f} \Phi(t_f - \tau)BB^T\Phi^T(t_f - \tau)d\tau R_f^{-1}x_f \\ &= \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau)d\tau R_f^{-1}x_f = x_f. \end{aligned} \tag{11}$$

Necessity. Let

$$p(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{12}$$

be the characteristic polynomial of the matrix $A \in M_n$. Then by the well-known [2] Cayley-Hamilton theorem we have

$$p(A) = A^n + a_{n-1}A^{n-1} + ... + a_1 A + a_0 I_n = 0. \tag{13}$$

Using (13) we may eliminate from (5) A^k for $k = n, n+1, ...$ and we obtain

$$\Phi(t) = \sum_{k=0}^{n-1} c_k(t)A^k, \tag{14}$$

where $c_k(t)$, $k = 0, 1, ..., n-1$ are some nonzero functions of time depending on the matrix A.

Substitution of (14) into

$$\int_0^{t_f} \Phi(t_f - \tau)Bu(\tau)d\tau \tag{15}$$

yields

$$x_f = \begin{bmatrix} B & AB & ... & A^{n-1}B \end{bmatrix} \begin{bmatrix} v_0(t_f) \\ v_1(t_f) \\ \vdots \\ v_{n-1}(t_f) \end{bmatrix}, \tag{16}$$

where

$$v_k(t_f) = \int_0^{t_f} c_k(\tau)u(t_f - \tau)d\tau, \quad k = 0, 1, \ldots, n-1. \tag{17}$$

For given $x_f \in \Re_+^n$ it is possible to find nonnegative $v_k(t_f)$ for $k = 0, 1, \ldots, n-1$ if and only if the matrix

$$\begin{bmatrix} B & AB & ... & A^{n-1}B \end{bmatrix} \tag{18}$$

has n linearly independent monomial columns and this takes place only if the matrix $[B, A]$ contains n linearly independent columns [9]. Note that for the nonnegative $v_k(t_f)$, $k = 0, 1, \ldots, n-1$ it is possible to find a nonnegative input $u(t) \in \Re_+^m$, $t \in [0, t_f]$ only if the matrix $B \in \Re_+^{n \times n}$ is monomial and the matrix $A \in M_n$ is diagonal.

Consider the fractional positive system (2) with $A \in M_n$ and $B \in \Re_+^{n \times m}$ monomial. If the system is reachable in time $t \in [0, t_f]$, then usually there exists many different inputs $u(t) \in \Re_+^n$ that steers the state of the system from $x_0 = 0$

to $x_f \in \Re_+^n$. Among these inputs we are looking for input $u(t) \in \Re_+^n$, $t \in [0, t_f]$ that minimizes the performance index

$$I(u) = \int_0^{t_f} u^T(\tau) Q u(\tau) d\tau, \tag{19}$$

where $Q \in \Re_+^{n\times n}$ is a symmetric positive defined matrix and $Q^{-1} \in \Re_+^{n\times n}$.

The minimum energy control problem for the fractional positive continuous-time linear systems (2) can be stated as follows.

Given the matrices $A \in M_n$, $B \in \Re_+^{n\times m}$, α and $Q \in \Re_+^{n\times n}$ of the performance matrix (19), $x_f \in \Re_+^n$ and $t > 0$, find an input $u(t) \in \Re_+^n$ for $t \in [0, t_f]$ that steers the state vector of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (19).

4. Problem solution

To solve the problem we define the matrix

$$\begin{aligned} W &= W(t_f, Q) \\ &= \int_0^{t_f} \Phi(t_f - \tau) B Q^{-1} B^T \Phi^T(t_f - \tau) d\tau, \end{aligned} \tag{20}$$

where $\Phi(t)$ is defined by (5). From (20) and Theorem 1 it follows that the matrix (20) is monomial if and only if the fractional positive system (2) is reachable in time $[0, t_f]$. In this case we may define the input

$$\hat{u}(t) = Q^{-1} B^T \Phi^T (t_f - t) W^{-1} x_f \quad \text{for} \quad \in [0, t_f]. \tag{21}$$

Note that the input (21) satisfies the condition $u(t) \in \Re_+^n$ for $t \in [0, t_f]$ if

$$Q^{-1} \in \Re_+^{n\times n} \quad \text{and} \quad W^{-1} \in \Re_+^{n\times n}. \tag{22}$$

Theorem 4. Let $\overline{u}(t) \in \Re_+^n$ for $t \in [0, t_f]$ be an input that steers the state of the fractional positive system (2) from $x_0 = 0$ to $x_f \in \Re_+^n$. Then the input (21) also steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (21), i.e. $I(\hat{u}) \le I(\overline{u})$.

The minimal value of the performance index (19) is equal to

$$I(\hat{u}) = x_f^T W^{-1} x_f. \tag{23}$$

Proof. If the conditions (22) are met then the input (21) is well defined and $\hat{u}(t) \in \Re_+^n$ for $t \in [0, t_f]$. We shall show that the input steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$. Substitution of (21) into (3) for $t = t_f$ and $x_0 = 0$ yields

$$\begin{aligned} x(t_f) &= \int_0^{t_f} \Phi(t_f - \tau) B \hat{u}(\tau) d\tau \\ &= \int_0^{t_f} \Phi(t_f - \tau) B Q^{-1} B^T \Phi^T (t_f - \tau) d\tau W_f^{-1} x_f = x_f \end{aligned} \tag{24}$$

since (20) holds. By assumption the inputs $\overline{u}(t)$ and $\hat{u}(t)$, $t \in [0, t_f]$ steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$. Hence

$$\begin{aligned} x_f &= \int_0^{t_f} \Phi(t_f - \tau) B \overline{u}(\tau) d\tau \\ &= \int_0^{t_f} \Phi(t_f - \tau) B \hat{u}(\tau) d\tau \end{aligned} \tag{25a}$$

or

$$\int_0^{t_f} \Phi(t_f - \tau) B [\overline{u}(\tau) - \hat{u}(\tau)] d\tau = 0. \tag{25b}$$

By transposition of (25b) and postmultiplication by $W^{-1} x_f$ we obtain

$$\int_0^{t_f} [\overline{u}(\tau) - \hat{u}(\tau)]^T B^T \Phi^T (t_f - \tau) d\tau W^{-1} x_f = 0. \tag{26}$$

Substitution of (21) into (26) yields

$$\begin{aligned} &\int_0^{t_f} [\overline{u}(\tau) - \hat{u}(\tau)]^T B^T \Phi^T (t_f - \tau) d\tau W^{-1} x_f \\ &= \int_0^{t_f} [\overline{u}(\tau) - \hat{u}(\tau)]^T Q \hat{u}(\tau) d\tau = 0. \end{aligned} \tag{27}$$

Using (27) it is easy to verify that

$$\begin{aligned} \int_0^{t_f} \overline{u}(\tau)^T Q \overline{u}(\tau) d\tau &= \int_0^{t_f} \hat{u}(\tau)^T Q \hat{u}(\tau) d\tau \\ &+ \int_0^{t_f} [\overline{u}(\tau) - \hat{u}(\tau)]^T Q [\overline{u}(\tau) - \hat{u}(\tau)] d\tau. \end{aligned} \tag{28}$$

From (28) it follows that $I(\hat{u}) < I(\overline{u})$ since the second term in the right-hand side of the inequality is nonnegative. To find the minimal value of the performance index (19) we substitute (21) into (19) and we obtain

$$\begin{aligned} I(\hat{u}) &= \int_0^{t_f} \hat{u}^T(\tau) Q u(\tau) d\tau \\ &= x_f^T W^{-1} \int_0^{t_f} \Phi(t_f - \tau) B Q^{-1} B^T \Phi^T (t_f - \tau) d\tau W^{-1} x_f \\ &= x_f^T W^{-1} x_f \end{aligned} \tag{29}$$

since (20) holds.

5. Procedure and example

Procedure.

Step 1. Using (5) compute the matrix $\Phi(t)$.

Step 2. Knowing the matrices A, B, Q and α, t_f using (20) compute the matrix W.
Step 3. Using (21) compute $\widehat{u}(t)$.
Step 4. Using (21) compute $I(\widehat{u})$.

Example. Consider the fractional positive system (2) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \alpha = 0.5 \tag{30}$$

and the performance index (19) with

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad t_f = 1.$$

Compute the optimal input $\widehat{u}(t)$, $t \in [0, 1]$ that steers the state of the system from $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and minimize the performance index.

Using Procedure we obtain the following:

Step 1. Using (5) and (30) we obtain

$$\begin{aligned} \Phi(t) &= \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^k \frac{t^{0.5(k-1)}}{\Gamma[0.5(k+1)]}. \end{aligned} \tag{31}$$

Step 2. From (20), (31) and (30) we have

$$\begin{aligned} W &= \int_0^1 \Phi(t_f - \tau) B Q^{-1} B^T \Phi^T (t_f - \tau) d\tau \\ &= \frac{1}{2} \int_0^1 \Phi^2 (1 - \tau) d\tau \end{aligned} \tag{32}$$

since

$$BQ^{-1}B^T = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\Phi(t_f - \tau) = \Phi^T (t_f - \tau)$$

.

Step 3. Using (21) and (32) we obtain

$$\begin{aligned} \widehat{u}(t) &= Q^{-1} B^T \Phi^T (t_f - t) W^{-1} x_f \\ &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Phi^T (1 - t) W^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \end{aligned} \tag{33}$$

where W is given by (32).

Step 4. The minimal value of the performance index is equal to

$$\begin{aligned} I(\widehat{u}) &= x_f^T W^{-1} x_f \\ &= [\ 1 \quad 1\ ] \left[\frac{1}{2} \int_0^1 \Phi^2 (1 - \tau) d\tau \right]^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \int_0^1 \Phi^2 (\tau) d\tau. \end{aligned} \tag{34}$$

6. Concluding remarks

The minimum energy control problem for the fractional positive continuous-time linear systems have been formulated and solved. Sufficient conditions for the existence of solution to the problem have been established (Theorem 4). A procedure for solving of the problem have been proposed and its effectiveness has been demonstrated on a numerical example. An open problem is an extension of these considerations to fractional positive descriptor continuous-time and discrete-time linear systems.

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